Cousin $\mathbb{A}^1$-motivic stable homotopy category.

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December 2020

Abstract

In this notes we outline the definition of the version of stable motivic homotopy category over a base S denoted $\mathbf{SH}^{\mathbb{A}^1}_{\mathrm{hf}}(S)$ that is a localisation of Morel-Voevodsky's stable $\mathbb{A}^1$-motivic homotopy category $\mathbf{SH}^{\mathbb{A}^1}(S)$ [9, 6, 7, 4] and treat some defects of $\mathbf{SH}^{\mathbb{A}^1}(S)$ over positive-dimensional schemes being equivalent to $\mathbf{SH}^{\mathbb{A}^1}(k)$ over fields. (1) Cousin complex of cohomology theories representable in $\mathbf{SH}^{\mathbb{A}^1}_{\mathrm{hf}}(S)$ is exact on local essentially smooth schemes, (2) sheaves of homotopy groups are strictly $\mathbb{A}^1$-invariant, (3) $\mathbf{SH}^{\mathbb{A}^1}_{\mathrm{hf}}(S)$ satisfies connectivity theorem. In the notes we give definitions and formulations; most of proofs are skipped.

Consider the ∞-category of (Nisnevich) sheaves of S^1-spectra $\mathrm{Sh}_{S^1}(S)$ on the category of smooth schemes Sm_S over a scheme S, that is the subcategory of the category of presheaves $\mathrm{Pre}_{S^1}(S)$ spanned by Nisnevich local objects, where $\mathrm{Pre}_{S^1}(S)$ is the category of radditive functors $\mathrm{Sm}_S \to \mathbf{SH}$. Consider the homotopy t-structure on $\mathrm{Sh}_{S^1}(S)$ and denote by $t_{\geq l}$ the truncation endofunctors, $t_{\leq l}\mathcal{F} = \mathrm{cofib}(\mathcal{F}_{\geq l} \to \mathcal{F})$. Then $t_{=l}\mathcal{F} = \mathrm{cofib}(t_{\geq l}\mathcal{F} \to t_{\geq l-1}\mathcal{F})$ is the Eilenberg-MacLane spectrum of the group sheaf $\underline{\pi}_l(\mathcal{F})$.

Definition 1. For each noetherian separated scheme S of finite Krull dimension define $\mathrm{Sh}_{\mathrm{hf},S^1}(S)$ as the maximal subcategory of $\mathrm{Sh}_{S^1}(S)$ such that for each closed immersion $i\colon Z \to S$ of codimension 1 with open complement $j\colon U = S - Z \to S$, the following holds:

(p) for any $\mathcal{F} \in \mathrm{Sh}_{\mathrm{hf},S^1}(S)$ the presheaf $i^!\mathcal{F}$ is in $\mathrm{Sh}_{\mathrm{hf},S^1}(Z)$, for any $\mathcal{F} \in \mathrm{Sh}_{\mathrm{hf},S^1}(U)$ the presheaf $j_*\mathcal{F}$ is in $\mathrm{Sh}_{\mathrm{hf},S^1}(S)$, and

(t) there are canonical equivalencs

$$i^!(t_{\leq l}\mathcal{F}) \to t_{\leq l}(i^!\mathcal{F}), \quad t_{\geq l}(j_*\mathcal{F}) \to j_*(t_{\geq l}\mathcal{F}).$$

Let us note that in the case of regular base scheme for the purposes described in what follows we can define a bigger category that definition is like as above and deals with the subclass of closed immersions of codimension one given by vanishing loci of regular function, or more generally by divisors.

Theorem 1. $\mathrm{Sh}_{\mathrm{hf},S^1}(S)$ *is the localising subcategory of* $\mathrm{Sh}_{S^1}(S)$.

Denote by $L_{\mathrm{hf}}\colon \mathrm{Sh}_{S^1}(S) \to \mathrm{Sh}_{\mathrm{hf},S^1}(S)$ the localisation functor left adjoint to the canonical embedding.

Denote $\mathrm{Sh}_{\mathrm{hf},S^1}(S)_{\leq l} = \mathrm{Sh}_{S^1}(S)_{\geq l} \cap \mathrm{Sh}_{\mathrm{hf},S^1}(S)$, and call it as subcategry of connective obejcts.

Theorem 2. *The functor* $\mathrm{Sh}_{\mathrm{hf},S^1}(S)$ *preserves connective objects.*

Theorem 3. *The functors* $t_{\leq l}$ *on* $\mathrm{Sh}_{S^1}(S)$ *preserves* $\mathrm{Sh}_{\mathrm{hf},S^1}(S)$.

Let $\mathrm{Sm}^{\mathrm{pair}}$ define the category with objects being pairs (X,U) given by $X \in \mathrm{Sm}_S$ and U being open subscheme in X, and morphisms being morphisms of pairs.

Definition 2. For any $\mathcal{F} \in \mathrm{Pre}(S)$, $X \in \mathrm{Sm}_S$, and closed subscheme Y in X consider the groups $A^i_Y(X) = [X/(X-Y), \mathcal{F}\wedge S^i]_{\mathbf{SH}^{\mathbb{A}^1}(S)} = \pi_{-i+1}\operatorname{cofib}(\mathcal{F}(X) \to \mathcal{F}(X-Y))$. This defines the functor $\mathrm{Sm}^{\mathrm{pair}} \to \mathrm{Ab}^{\mathbb{Z}}$ form the category of smooth open pairs to graded abelian groups. We call presheaves A^* cohomology theory defined by presheaf $\mathcal{F}$. The functors A^* extends by continuity on pairs of essentially smooth schemes.

Theorem 4. *For any and closed immersion* $T \subset S$ *and closed immersion* $Z \to T$ *of codimension 1 with open complement* $U = S - Z$*, for any essentially smooth local scheme* X *over* S*, and* $\mathcal{F} \in \mathrm{Sh}_{\mathrm{hf},S^1}(S)$ *the sequence*

$$0 \to A^i_T(X) \to A^i_{T-Z}(X - X_Z) \to A^{i+1}_Z(X) \to 0,$$

where A^i *are given by Definition 2, is exact.*

Proof. Let $\mathcal{F}' = r_*\mathcal{F}$. Reduce to the case $S = T$.

The functors i_* and j^* commutes with functors $t_{=l}$. Since j^* and i_* preserves $\mathrm{Sh}_{\mathrm{hf},S^1}(S)$ and $i^!(-)[1]$ and j_* commutes with $l_{=l}$ on $\mathrm{Sh}_{\mathrm{hf},S^1}(S)$ the sequence

$$\to t_{=l}\mathcal{F} \to t_{=l}j_*j^*\mathcal{F} \to t_{=l}i_*i^!\mathcal{F}[1] \to \dots$$

is distinguished triangle. So the sequence

$$\pi_l\mathcal{F} \to \pi_l j_*j^*\mathcal{F} \to \pi_{l-1}i_*i^!\mathcal{F}$$

is short exact. Now the claim follows

$$j_*j^*\mathcal{F}(X) \simeq \mathcal{F}(X-X_Z), i_*i^!\mathcal{F}[1](X) \simeq i^!\mathcal{F}[1](X_Z) \simeq \operatorname{cofib}(\mathcal{F}(X) \to \mathcal{F}(X-X_Z)).$$

□

Corollary 1. *Let* $d = \dim S$*, then for* A^* *defined by* $\mathcal{F} \in \mathrm{Sh}_{\mathrm{hf},S^1}(S)$*, see Definition 2, and essentially smooth local* X *over* S *the sequence*

$$A^0(X) \to \coprod_{\eta\in S^{(0)}} A^0(\eta) \to \coprod_{z\in S^{(1)}} A^1_z(X) \to \cdots \coprod_{z\in S^{(i)}} A^i_z(X) \to \cdots \coprod_{z\in S^{(d)}} A^d_z(X),$$

where $A^i_z(X) = A^i_Y(X)$*,* $Y = X \times_S z$*, and* $S^{(l)}$ *denotes the set of points of codimension* l *in* S*, is exact.*

Definition 3. $\mathbf{SH}^{\mathbb{A}^1}_{\mathrm{hf},S^1}(S) = \mathrm{Sh}_{\mathrm{hf},S^1}(S) \cap \mathbf{SH}^{\mathbb{A}^1}_{S^1}(S)$.

In the following theorem we use the similar argument as in [10].

Theorem 5. *The Nisnevich sheaf $\underline{\pi}_i\mathcal{F}$ associated with the presheaf of stable homotopy groups $\pi_i\mathcal{F}$ is strict $\mathbb{A}^1$-invariant for any $\mathcal{F} \in \mathbf{SH}^{\mathbb{A}^1}_{\mathrm{hf},S^1}$ $i \in \mathbb{Z}$.*

Proof. Without loss of generality we can assume $i = 0$. Since $\mathcal{F} \in \mathbf{SH}^{\mathbb{A}^1}_{\mathrm{hf},S^1}(S)$ by Corollary 1

$$\underline{\pi}_0\mathcal{F}(X) \to \coprod_{\eta \in S^{(0)}} A^0(\eta) \to \coprod_{z \in S^{(1)}} A^1_z(X) \to \cdots \coprod_{z \in S^{(i)}} A^i_z(X) \to \cdots \coprod_{z \in S^{(d)}} A^d_z(X) \tag{1}$$

gives exact sequence of Nisnevich sheaves. Since all the sheaves $X \mapsto \coprod_{z \in S^{(i)}} A^i_z(X)$ are flasque sequence (1) gives a flasque resolvent of the Nisnevich sheaves $\underline{\pi}_0\mathcal{F}$. Since $\mathcal{F}$ is $\mathbb{A}^1$-invariant $\coprod_{z \in S^{(i)}} A^i_z(X) \simeq \coprod_{z \in S^{(i)}} A^i_z(X \times \mathbb{A}^1)$ and hence the resolvent of $\underline{\pi}_0\mathcal{F}$ given by (1) is $\mathbb{A}^1$-invariant. Thus the Nisnevich cohomologies of $\underline{\pi}_0\mathcal{F}$ are $\mathbb{A}^1$-invariant. □

Theorem 6. *The functor $\mathrm{Sh}_{\mathrm{hf},S^1}(S) \to \mathbf{SH}^{\mathbb{A}^1}_{\mathrm{hf},S^1}(S)$ preserves connective objects.*

Proof for the case of one-dimensional scheme S. Let X be local essentially smooth schemes over S. Let e be the generic point of the closed fibre of X, let η be the generic point of X. It follows by [2] that $\pi_i(\mathcal{F})(X) \hookrightarrow \pi_i(\mathcal{F})(X_e)$. By Corollary 1 $\pi_i(\mathcal{F})(X_e) \hookrightarrow \pi_i(\mathcal{F})(\eta)$. Now the claim follows by [8, 11] since $\dim \eta = 0$. □

Theorem 7. *The functor L_{hf} commutes with $L_{\mathbb{A}^1}$ and $L_{\mathbb{A}^1,\mathbb{G}_m}$ preserves $\mathbb{A}^1$-invariant framed presheaves of S^1-spectra and $\Omega_{\mathbb{G}_m}$-bi-spectra of $\mathbb{A}^1$-invariant framed presheaves.*

The generalisation of the the central result of [5] hold.

Theorem 8. *There is a simplicial homotopy equivalence*

$$L_{\mathrm{mothf}} \simeq L_{\mathrm{hfnis}}(L_{\mathbb{A}^1}\mathrm{Fr}(\Sigma_{S^1 \wedge \mathbb{G}_m} X))^{\mathrm{gp}},$$

there L_{mothf} denotes localisation endofunctor given by the composition $\mathrm{Spt}_{S^1,\mathbb{G}_m}(S) \to \mathbf{SH}^{\mathbb{A}^1}_{\mathrm{hf},S^1,\mathbb{G}_m}(S) \to \mathrm{Spt}_{S^1,\mathbb{G}_m}(S)$.

In some since the above result is an improvement of the result of [3, 1] though we do not mean that that above theorem recovers and implies the mentioned result, since the assumption on $\mathcal{F}$ is stronger.

Definition 4. For each noetherian separated schemes S of finite Krull dimension define $\mathrm{Sh}_{\mathrm{hf},S^1,\mathrm{strong}}(S)$ as the maximal subcategory of $\mathrm{Sh}_{S^1}(S)$ such that for each $\mathcal{F} \in \mathrm{Sh}_{\mathrm{hf},S^1}(S)$ and closed immersion $i\colon Y \to X$ of codimension 1, $i^!\mathcal{F} \in \mathrm{Sh}_{\mathrm{hf},S^1}(Z)$ and there is a canonical equivalence

$$i^! t_{\leq l}\mathcal{F}\big|_X \to t_{\leq l} i^! \mathcal{F}, t_{\geq l} j_* \mathcal{F} \to j_* t_{\geq l}\mathcal{F}.$$

$\mathcal{F}\big|_X$ denotes the restriction of $\mathcal{F}$ on the small étale site Et_X over X.

Define $\mathbf{SH}^{\mathbb{A}^1}_{\mathrm{hf},S^1,\mathrm{strong}} = \mathrm{Sh}_{\mathrm{hf},S^1,\mathrm{strong}}(S) \cap \mathbf{SH}^{\mathbb{A}^1}(S)$.

Then the argument of Corollary 1, Theorem 5, and Theorem 6 implies the for any noetherian spaerted base scheme S of finite Krull dimension

Theorem 9. *Let $d = \dim S$, then for A^* defined by $\mathcal{F} \in \mathrm{Sh}_{\mathrm{hf},S^1,\mathrm{strong}}(S)$, and essentially smooth local X over S the sequence*

$$A^0(X) \to \coprod_{\eta \in S^{(0)}} A^0(\eta) \to \coprod_{z \in S^{(1)}} A^1_z(X) \to \cdots \coprod_{z \in S^{(i)}} A^i_z(X) \to \cdots \coprod_{z \in S^{(d)}} A^d_z(X),$$

where $A^i_z(X) = A^i_Y(X)$, $Y = X \times_S z$, is exact.

Theorem 10. *The Nisnevich sheaf $\underline{\pi}_i\mathcal{F}$ associated with the presheaf of stable homotopy groups $\pi_i\mathcal{F}$ is strict $\mathbb{A}^1$-invariant for any $\mathcal{F} \in \mathbf{SH}^{\mathbb{A}^1}_{\mathrm{hf},S^1,\mathrm{strong}}$ $i \in \mathbb{Z}$.*

Theorem 11. *The functor $\mathrm{Sh}_{\mathrm{hf},S^1,\mathrm{strong}} \to \mathbf{SH}_{\mathrm{hf},S^1,\mathrm{strong}}$ preserves connective objects.*

Conjecture 1. *The canonical functor $\mathbf{SH}^{\mathbb{A}^1}_{\mathrm{hf},S^1,\mathrm{strong}} \to \mathbf{SH}_{\mathrm{hf},S^1}$ is an equivalence.*

References

[1] Druzhinin. *Nisnevich localisation of tf-motivic framed spectra, in preparation.*

[2] Andrei Druzhinin. *Stable connectivity over a base.* arXiv: `1911.05014`.

[3] Andrei Druzhinin, Håkon Kolderup, and Paul Arne Østvær. *Strict $\mathbb{A}^1$-invariance over the integers.* arXiv: `arxiv:2012.07365`.

[4] B. I. Dundas, O. Röndigs, and P. A. Østvær. "Motivic Functors". In: *Documenta Math* 8 (2003), pp. 489–526.

[5] Grigory Garkusha and Ivan Panin. *Framed motives of algebraic varieties (after V. Voevodsky).* To appear in *J. Am. Math. Soc.* 2018. arXiv: `1409.4372`.

[6] J. F. Jardine. "Motivic symmetric spectra". In: *Doc. Math.* 5 (2000), 445–553 (electronic). ISSN: 1431-0635.

[7] Fabien Morel. "An introduction to $\mathbb{A}^1$-homotopy theory". In: *Contemporary developments in algebraic K-theory.* ICTP Lect. Notes, XV. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004, pp. 357–441.

[8] Fabien Morel. "The stable $\mathbb{A}^1$-connectivity theorems". In: *K-Theory* 35.1-2 (2005), pp. 1–68. ISSN: 0920-3036. DOI: `10.1007/s10977-005-1562-7`. URL: `http://dx.doi.org/10.1007/s10977-005-1562-7`.

[9] Fabien Morel and Vladimir Voevodsky. "$\mathbb{A}^1$-homotopy theory of schemes". In: *Inst. Hautes Études Sci. Publ. Math.* 90 (1999), 45–143 (2001). ISSN: 0073-8301. URL: `http://www.numdam.org/item?id=PMIHES_1999__90__45_0`.

[10] I. A. Panin. "Nice triples and moving lemmas for motivic spaces". In: *Izv. Math.* 83.4 (2019), pp. 796–829.

[11] Johannes Schmidt and Florian Strunk. "Stable $\mathbb{A}^1$-connectivity over Dedekind schemes". In: *Annals of K-Theory* 3.2 (March 2018), pp. 331–367. DOI: `10.2140/akt.2018.3.331`.