Latala's inequality

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April 25, 2015

1 Formulation and a simplification

Let $X_1, X_2, \ldots, X_n$ be i.i.d non-negative random variables with the distribution X. We suppose that $\mathbb{E}\, X = 1$ and $\mathbb{E}\, |X-1| = \alpha \neq 0$. Define their products

$$R_0 = 1; \quad R_k = \prod_{j=1}^{k} X_j,\ k = 1, 2, \ldots, n.$$

The following theorem is proved in [2] (see also [1] for the $p < 1$ version).

Theorem 1 (Latala's theorem)**.** *The inequality*

$$\sum_{j=0}^{n} \|v_j\| \lesssim \mathbb{E}\, \Big\| \sum_{j=0}^{n} v_j R_j \Big\| \leqslant \sum_{j=0}^{n} \|v_j\|$$

holds independently of the norm $\|\cdot\|$ and n (the constant depends only on α).

Here the v_j are vectors in some Banach space $\mathbb{B}$. Surely, only the left hand-side inequality is interesting. Latala's theorem is more general, we have stated a particular case (in the original version, the random variables X_j may have different distributions; however, the discussed case already contains all the difficulties). Our aim here is to explain the proof and provide a small simplification (which will lead to some drop of the constant in the main inequality, but, maybe, make the proof more transparent for an analyst).

Lemma 1. *Consider the i.i.d random variables Y_j such that $Y_j = 1 \pm \alpha$ with probability $\frac{1}{2}$. Define $\tilde{R}_j$ to be their products. Then,*

$$\mathbb{E}\, \Big\| \sum_{j=0}^{n} v_j \tilde{R}_j \Big\| \leqslant \mathbb{E}\, \Big\| \sum_{j=0}^{n} v_j R_j \Big\|.$$

Proof. We may suppose that there exist independent events E_j such that $\mathbb{E}(X_j \mid E_j) = 1+\alpha$ and $P(E_j) = \frac{1}{2}$ (to do this, represent the probability space as an infinite direct product of unit intervals such that each X_j depends on the jth coordinate only). Let S be the algebra of sets generated by the E_j. Then, by Jensen's inequality:

$$\mathbb{E}\left(\Big\| \sum_{j=0}^{n} v_j R_j \Big\| \;\Big|\; S \right) \leqslant \Big\| \sum_{j=0}^{n} v_j R_j \Big\|$$

almost surely. It remains to notice that the expression on the left coincides with

$$\Big\| \sum_{j=0}^{n} v_j \tilde{R}_j \Big\|,$$

if Y_j equals $1+\alpha$ on E_j and $1-\alpha$ on the complement of E_j. $\square$

This lemma shows that we may work with the case $X_j = Y_j$. It may seem convenient to work with the Rademacher representation of the Y_j and proceed analytically. However, the author does not know how to prove Theorem 1 in this way.

The proof will be by induction in n, we will drop v_0, use the induction assumption, and then try to get a bound for the sum with v_0 present there. However, this will be rather delicate. There will be two principles. The first one is the statement of the theorem for $n = 1$ (one can estimate $\|v_0\|$ and $\|v_1\|$ by $\mathbb{E}\,\|v_0 + X_1 v_1\|$), the second one is that if one adds a big constant vector to a random vector, then the mathematical expectation of the norm of the latter increases in a reasonable way.

2 Preliminary lemmas

We begin with the variant of the case $n = 2$ of Theorem 1. One can easily see that these inequalities are sharp (take $\mathbb{B} = \mathbb{R}$).

Lemma 2. *For any $v_0, v_1 \in \mathbb{B}$,*

$$\mathbb{E}\,\big\|v_0 + X_1 v_1\big\| \geqslant \alpha\|v_1\|; \quad \mathbb{E}\,\big\|v_0 + X_1 v_1\big\| \geqslant \frac{\alpha}{\alpha+1}\|v_0\|.$$

Proof. The first inequality may be rewritten as

$$\frac{1}{2}\big\|v_0 + (1+\alpha)v_1\big\| + \frac{1}{2}\big\|v_0 + (1-\alpha)v_1\big\| \geqslant \alpha\|v_1\|,$$

which is nothing but the triangle inequality. The second inequality also follows from the triangle inequality:

$$\frac{1}{2}\big\|v_0 + (1+\alpha)v_1\big\| + \frac{1}{2}\big\|v_0 + (1-\alpha)v_1\big\| = \frac{1+\alpha}{2}\Big\|\frac{v_0}{1+\alpha} + v_1\Big\| + \frac{1-\alpha}{2}\Big\|\frac{v_0}{1-\alpha} + v_1\Big\| \geqslant$$
$$\frac{1-\alpha}{2}\Big\|\frac{v_0}{1+\alpha} + v_1\Big\| + \frac{1-\alpha}{2}\Big\|\frac{v_0}{1-\alpha} + v_1\Big\| \geqslant \frac{1-\alpha}{2}\Big(\frac{1}{1-\alpha} - \frac{1}{1+\alpha}\Big)\|v_0\| \geqslant \frac{\alpha}{1+\alpha}\|v_0\|.$$

□

Lemma 3. *Let Y be a random vector in $\mathbb{B}$. Suppose $v \in \mathbb{B}$ be such that $P(\|Y\| \geqslant \frac{\|v_0\|}{4}) \leqslant \frac{1}{4}$. Then,*

$$\mathbb{E}\,\big\|Y + v\big\| \geqslant \mathbb{E}\,\|Y\| + \frac{\|v_0\|}{8}.$$

Proof. We use Bayes's formula and the triangle inequality:

$$\mathbb{E}\,\big\|Y + v\big\| = P\big(\|Y\| \geqslant \frac{\|v_0\|}{4}\big)\,\mathbb{E}\,\Big(\big\|Y+v\big\| \,\Big|\, \|Y\| \geqslant \frac{\|v_0\|}{4}\Big) + P\big(\|Y\| \leqslant \frac{\|v_0\|}{4}\big)\,\mathbb{E}\,\Big(\big\|Y+v\big\| \,\Big|\, \|Y\| \leqslant \frac{\|v_0\|}{4}\Big) \geqslant$$
$$P\big(\|Y\| \geqslant \frac{\|v_0\|}{4}\big)\Big(\mathbb{E}\,\Big(\big\|Y\big\| \,\Big|\, \|Y\| \geqslant \frac{\|v_0\|}{4}\Big) - \|v_0\|\Big) + P\big(\|Y\| \leqslant \frac{\|v_0\|}{4}\big)\Big(\frac{\|v_0\|}{2} + \mathbb{E}\,\Big(\big\|Y\big\| \,\Big|\, \|Y\| \leqslant \frac{\|v_0\|}{4}\Big)\Big) \geqslant$$
$$\mathbb{E}\,\|Y\| + \Big(\frac{P\big(\|Y\| \leqslant \frac{\|v_0\|}{4}\big)}{2} - P\big(\|Y\| \geqslant \frac{\|v_0\|}{4}\big)\Big)\|v_0\| \geqslant \mathbb{E}\,\|Y\| + \frac{\|v_0\|}{8}.$$

We used the inequality $\|Y + v_0\| \geqslant \|v_0\| - \|Y\| \geqslant \frac{\|v_0\|}{2} + \|Y\|$ for the second summand, when we were passing from the first line to the second one. □

This lemma is a formalization of our second principle that says that when perturbs a random vector with a big constant vector, the average norm increases.

3 First attempts

Suppose that we are going to prove the inequality

$$c\sum_{j=0}^{n}\|v_j\| \leqslant \mathbb{E}\Big\|\sum_{j=0}^{n} v_jR_j\Big\|, \tag{1}$$

where c is sufficiently small constant, by induction in n. Suppose that it holds for $n-1$. For that we write

$$\mathbb{E}\Big\|\sum_{j=0}^{n} v_jR_j\Big\| = \frac{1}{2}\,\mathbb{E}\Big\|v_0+(1-\alpha)v_1+(1-\alpha)\sum_{j=2}^{n} v_j\prod_{k=1}^{j}X_k\Big\|+\frac{1}{2}\,\mathbb{E}\Big\|v_0+(1+\alpha)v_1+(1+\alpha)\sum_{j=2}^{n} v_j\prod_{k=1}^{j}X_k\Big\| \tag{2}$$

and apply the induction hypothesis to each of the summands (with vectors $v_0+(1-\alpha)v_1$, $\{(1-\alpha)v_j\}_{j=2}^{n}$ and variables $X_2, X_3, \dots, X_n$ in the first summand and vectors $v_0+(1+\alpha)v_1$, $\{(1+\alpha)v_j\}_{j=2}^{n}$ and variables $X_2, X_3, \dots, X_n$ in the second). This will lead us to

$$\mathbb{E}\Big\|\sum_{j=0}^{n} v_jR_j\Big\| \geqslant c\Big(\frac{1}{2}\|v_0+(1-\alpha)v_1\|+\frac{1}{2}\|v_0+(1+\alpha)v_1\|\Big)+\sum_{j=2}^{n} c\|v_j\|.$$

So, we will have to bound $c\|v_0\|+c\|v_1\|$ by

$$c\Big(\frac{1}{2}\|v_0+(1-\alpha)v_1\|+\frac{1}{2}\|v_0+(1+\alpha)v_1\|\Big).$$

It is impossible. However, this reasoning gives us hope that if we pass from inequality (1) to a more general one, the induction may work. Let us try to prove the inequality

$$\mathbb{E}\Big\|\sum_{j=0}^{n} v_jR_j\Big\| \geqslant \sum_{j=0}^{n} c_j\|v_j\|, \tag{3}$$

where c_j are some coefficients that are uniformly bounded from below. Of course, inequalities (3) and (1) are equivalent, but (3) seems to be more "induction-friendly". Indeed, let us try to do the same reasoning again (i.e., use formula (2) and apply the induction hypothesis (3) to the same vectors and random variables). This will lead us to the inequality

$$\mathbb{E}\Big\|\sum_{j=0}^{n} v_jR_j\Big\| \geqslant c_0\Big(\frac{1}{2}\|v_0+(1-\alpha)v_1\|+\frac{1}{2}\|v_0+(1+\alpha)v_1\|\Big)+\sum_{j=2}^{n} c_{j-1}\|v_j\|.$$

So, to deduce inequality (3) for $n-1$ summands, we have to show that

$$c_0\Big(\frac{1}{2}\|v_0+(1-\alpha)v_1\|+\frac{1}{2}\|v_0+(1+\alpha)v_1\|\Big)+\sum_{j=2}^{n}\big(c_{j-1}-c_j\big)\|v_j\| \geqslant c_0\|v_0\|+c_1\|v_1\|.$$

This seems to be not as hopeless as the previous attempt if we make a proper choice of the c_j (however, this inequality also seems to be untrue if v_0 is big, because it turns into an equality when $v_0\to\infty$; but for this case we have the second principle, Lemma 3). It seems reasonable to take $c_j-c_{j-1}\geqslant 0$. Again, we use Lemma 2 to estimate the sum of linear combinations on the left hand-side:

$$c_0\Big(\frac{1}{2}\|v_0+(1-\alpha)v_1\|+\frac{1}{2}\|v_0+(1+\alpha)v_1\|\Big)+\sum_{j=2}^{n}\big(c_{j-1}-c_j\big)\|v_j\| \geqslant c_0\alpha\|v_1\|+\sum_{j=2}^{n}\big(c_{j-1}-c_j\big)\|v_j\|.$$

So, we can make prove the induction step when

$$c_0\|v_0\| \leqslant (c_0\alpha - c_1)\|v_1\| + \sum_{j=2}^{n}\big(c_{j-1} - c_j\big)\|v_j\|. \tag{4}$$

Again, it is convenient to assume that $c_0\alpha > c_1$.

How can we deal with the case when condition (4) is not satisfied? We are going to apply Lemma 3 with the constant vector v_0 and $Y = \sum_{j=1}^{n} v_j R_j$ in the role of the random vector. Therefore, we will need to estimate the probability of the event $\|Y\| \geqslant \frac{1}{4}\|v_0\|$. Of course, we will have to use the fact that inequlity (4) does not hold (to link Y and v_0). So, we will have to estimate the probability of the event

$$|X_1|\sum_{j=1}^{n} v_j \prod_{k=1}^{j} X_k \geqslant (c_0\alpha - c_1)\|v_1\| + \sum_{j=2}^{n}\big(c_{j-1} - c_j\big)\|v_j\|. \tag{5}$$

4 Tails estimates

A brief examination of our assumptions show that the value $c_j - c_{j-1}$ tends to zero as $j \to \infty$. Therefore, we cannot derive any information about the event (5) from the trivial estimate $\mathbb{E}\,\|\sum_{j=1}^{n} v_j R_j\| \leqslant \sum \|v_j\|$. However, we have not used the assumption that the X_j are independent seriously yet. It appears, that to get the desired estimates, we have to leave the convex world.

Lemma 4. *Suppose the variables X_j be as above. Then,*

$$\Big(\mathbb{E}\Big\|\sum_{j=0}^{n} v_j R_j\Big\|^{\frac{1}{2}}\Big)^2 \leqslant \frac{1}{1-\beta}\sum_{j=0}^{n}\beta^j\|v_j\|.$$

for some $\beta < 1$.

Proof. We use the inequality $(\sum |x_k|)^{\frac{1}{2}} \leqslant \sum |x_k|^{\frac{1}{2}}$ and the independence of the X_j:

$$\mathbb{E}\Big\|\sum_{j=0}^{n} v_j R_j\Big\|^{\frac{1}{2}} \leqslant \sum_{j=0}^{n}\mathbb{E}\Big\|v_j R_j\Big\|^{\frac{1}{2}} = \sum_{j=0}^{n}\|v_j\|^{\frac{1}{2}}\Big(\mathbb{E}\,|X|^{\frac{1}{2}}\Big)^j = \sum_{j=0}^{n}\|v_j\|^{\frac{1}{2}}\Big(\frac{\sqrt{1+\alpha}+\sqrt{1-\alpha}}{2}\Big)^j \leqslant \sum_{j=0}^{n}\|v_j\|^{\frac{1}{2}}\beta^j.$$

Here $\beta = \frac{\sqrt{1+\alpha}+\sqrt{1-\alpha}}{2} < 1$. It remains to use the Cauchy inequality:

$$\sum_{j=0}^{n}\|v_j\|^{\frac{1}{2}}\beta^j \leqslant \Big(\sum_{j=0}^{n}\beta^j\Big)^{\frac{1}{2}}\Big(\sum_{j=0}^{n}\beta^j\|v_j\|\Big)^{\frac{1}{2}} = \Big(\frac{1}{1-\beta}\Big)^{\frac{1}{2}}\Big(\sum_{j=0}^{n}\beta^j\|v_j\|\Big)^{\frac{1}{2}}.$$

□

By Chebyshev's inequality, Lemma 4 leads to

$$P\Big(\Big\|\sum_{j=0}^{n} v_j R_j\Big\| \geqslant \frac{t}{1-\beta}\sum_{j=0}^{n}\beta^j\|v_j\|\Big) \leqslant \frac{1}{\sqrt{t}}. \tag{6}$$

There are two things to be noticed. First, it is reasonable to take $c_j - c_{j-1} = c\beta^j$ of something like that. Second, the event from formula (5) differs from the one estimated in (6) (we have to multiply by X_1). This gives us some more freedom. Indeed, with probability $\frac{1}{2}$, X_1 is smaller than one, which allows to use estimate (6) with bigger t.

5 Final reasoning

We take N to be a big natural number. We define the coefficients c_j by the rule

$$c_0 = a; \quad c_1 = c_2 = \ldots = c_N = b; \quad c_j = b - \frac{d\sum_{k=N+1}^{j}\beta^k}{1-\beta}, j \geqslant N+1.$$

Here a, b, and d are small positive constants to be specified later. In order the c_j satisfy all the above requirements (they are $c_0 > \alpha c_1$ and $c_j > 0$ uniformly), these parameters must be under the conditions

$$b < a\alpha, \quad d < b.$$

We assume inequality (3) for $n-1$ and try to prove its version for n. It remains to deal with the case when inequality (4) does not hold, which turns into

$$\|v_0\| \geqslant \big(\alpha - \frac{b}{a}\big)\|v_1\| + \frac{d}{a(1-\beta)}\sum_{j=N+1}^{n}\beta^{(j-N-1)}\|v_j\|. \tag{7}$$

Consider the event Ω:

$$\Omega = \{X_2 = X_3 = \ldots = X_N = 1-\alpha\},$$

we have left X_1 free for further use. Let $\tilde{X}_j$ be X_j conditioned on Ω. Then,

$$\mathbb{E}\,\|\sum_{j=0}^{n} v_j R_j\| = \frac{1}{2}\,\mathbb{E}\,\Big\|w_0^- + (1-\alpha)^N \sum_{j=N+1}^{n} v_j \prod_{k=N+1}^{j}\tilde{X}_j\Big\| + \frac{1}{2}\,\mathbb{E}\,\Big\|w_0^+ + (1+\alpha)(1-\alpha)^{N-1}\sum_{j=N+1}^{n} v_j \prod_{k=N+1}^{j}\tilde{X}_j\Big\|,$$

where

$$w_0^- = v_0 + (1-\alpha)\Big(\sum_{j=1}^{N}(1-\alpha)^j v_j\Big); \quad v_0 + (1+\alpha)\Big(\sum_{j=1}^{N}(1-\alpha)^j v_j\Big).$$

By Lemma 2, $\|w_0^-\| + \|w_0^+\| \geqslant \frac{\alpha}{\alpha+1}\|v_0\|$, so at least one of these vectors is bigger than $\frac{\alpha}{2(\alpha+1)}\|v_0\|$. In order to apply Lemma 3, we need to estimate the probability of the event

$$\frac{\alpha}{2(\alpha+1)}\|v_0\| \leqslant (1+\alpha)(1-\alpha)^{N-1}\sum_{j=N+1}^{n} v_j \prod_{k=N+1}^{j}\tilde{X}_j.$$

By assumption (7), this event leads to (we also killed the summand with $\|v_1\|$)

$$\sum_{j=N+1}^{n} v_j \prod_{k=N+1}^{j}\tilde{X}_j \geqslant \frac{\alpha d}{2(1+\alpha)^2(1-\alpha)^{N-1}a(1-\beta)}\Big(\sum_{j=N+1}^{n}\beta^{(j-N-1)}\|v_j\|\Big).$$

By inequality (6), the probability of this event does not exceed

$$\sqrt{\frac{2(1+\alpha)^2(1-\alpha)^{N-1}a}{\alpha d}}.$$

So, we require

$$\sqrt{\frac{2(1+\alpha)^2(1-\alpha)^{N-1}a}{\alpha d}} < \frac{1}{4},$$

which seems to be true when N is sufficiently big. So, we can use Lemma 3 for the bigger of the vectors w_0^+ and w_0^- to get

$$\mathbb{E}\,\Big\|w_0^{\pm} + (1\pm\alpha)(1-\alpha)^{N-1}\sum_{j=N+1}^{n} v_j \prod_{k=N+1}^{j}\tilde{X}_j\Big\| \geqslant \frac{\alpha}{8(1+\alpha)}\|v_0\| + \mathbb{E}\,\Big\|(1\pm\alpha)(1-\alpha)^{N-1}\sum_{j=N+1}^{n} v_j \prod_{k=N+1}^{j}\tilde{X}_j\Big\|.$$

References

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[2] R. Latala, *L_1-norm of combinations of products of independent random variables*, Israel J. Math. **203** (2014), 295–308.