A note on approximation of analytic Lipschitz functions on strips and semi-strips

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Abstract

We give an alternative proof of the central lemma in [1] and provide a slight generalization.

1 The first problem

We investigate analytic functions on strips. We denote the real and imaginary parts of z by $\Re z$ and $\Im z$ respectively. For any $\kappa \geqslant 0$, let Π_κ be the strip of width 2κ:

$$\Pi_\kappa = \{z \in \mathbb{C} \mid |\Im z| \leqslant \kappa\}$$

and let Π_κ^+ be the semi-strip of the same width

$$\Pi_\kappa = \{z \in \mathbb{C} \mid |\Im z| \leqslant \kappa,\ \Re z \geqslant 0\}.$$

We start with a reformulation of Lemma 3.1 in [1].

Lemma 1.1. *For any $0 < \beta < \gamma$ and any $\varepsilon > 0$, there exists a number $C = C(\varepsilon, \beta, \gamma)$ with the following property. For any analytic function $U \colon \Pi_\beta^+ \to \mathbb{C}$ there exists an analytic function $V \colon \Pi_\gamma^+ \to \mathbb{C}$ such that*

$$\|V - U\|_{L_\infty(\Pi_0^+)} \leqslant \varepsilon \quad \textit{and} \quad \|V\|_{\mathrm{Lip}(\Pi_\gamma^+)} \leqslant C\|U\|_{\mathrm{Lip}(\Pi_\beta^+)}. \tag{1.1}$$

In [1], the authors used analytic partition of unity and the Jackson–Bernstein theorem to prove Lemma 1.1. We present another approach and start with a slight generalization.

Lemma 1.2. *For any Lipschitz function $f \colon \mathbb{R}_+ \to \mathbb{C}$, any $\gamma > 0$, and any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, \gamma)$ and an entire function V such that*

$$\|f - V\|_{L_\infty(\Pi_0^+)} \leqslant \varepsilon \quad \textit{and} \quad \|V\|_{\mathrm{Lip}(\Pi_\gamma^+)} \leqslant C\|f\|_{\mathrm{Lip}(\mathbb{R}_+)}. \tag{1.2}$$

Proof. Let $\tilde{f}$ be any Lipschitz extension of f to the whole real line. Fix a Schwartz function χ on the line with spectrum in $[-1, 1]$ and unit integral. Define f_δ by the formula

$$f_\delta = \tilde{f} * \chi_\delta, \qquad \chi_\delta(x) = \delta^{-1}\chi(\delta^{-1}x).$$

The spectrum of f_δ belongs to $[-\delta^{-1}, \delta^{-1}]$. By the Paley–Wiener–Schwartz theorem, f_δ extends to the entire function V of type δ^{-1}. We shall prove that V is the function we are looking for provided δ is sufficiently small. Let us prove the first property:

$$\Big|\tilde{f} - \tilde{f} * \chi_\delta\Big|(x) = \Big|\int (\tilde{f}(x) - \tilde{f}(x-y))\chi\Big(\frac{y}{\delta}\Big)\,d\frac{y}{\delta}\Big| \lesssim \|\tilde{f}\|_{\mathrm{Lip}} \int |y|\Big|\chi\Big(\frac{y}{\delta}\Big)\Big|\,d\frac{y}{\delta} \lesssim \|f\|_{\mathrm{Lip}} \int |\delta y||\chi(y)|\,dy = O(\delta)\|f\|_{\mathrm{Lip}}.$$

So, the first property is satisfied if we take $\delta = K\varepsilon$ for sufficiently small constant K. Let us verify the second one. We have to estimate $\|\partial V\|_{L_\infty(\Pi_\gamma^+)}$. We express $\partial V(\cdot, y)$ in terms of $\tilde{f}'$:

$$\partial V(z) = \int_{\mathbb{R}} 2\pi i\xi \hat{f}_\delta(\xi) e^{2\pi i\xi z}\, d\xi = \int_{\mathbb{R}} 2\pi i\xi \hat{\chi}(\delta\xi)\hat{\tilde{f}}(\xi) e^{2\pi i\xi(x+iy)}\, d\xi, \qquad z = x + iy.$$

Thus, $\partial V(\cdot, y)$ can be expressed as $M_y[\tilde{f}']$, where M_y is the Fourier multiplier with the symbol $\hat{\chi}(\delta\xi)e^{-2\pi y\xi}$. It suffices to prove that this multiplier acts on L_∞ with uniformly bounded norm when y is bounded. This is trivial since the symbol $\hat{\chi}(\delta\xi)e^{-2\pi y\xi}$ is uniformly bounded in any Schwartz semi-norm. $\square$

2 Extensions of analytic functions

Lemma 2.1. *For any $0 < \beta < \gamma$, any $\varepsilon > 0$, and any analytic function $U\colon \Pi_\beta \to \mathbb{C}$, there exists an analytic function $V\colon \Pi_\gamma \to \mathbb{C}$ such that*

$$\|U - V\|_{L_\infty(\Pi_\beta)} \leqslant \varepsilon \quad \text{and} \quad \|V\|_{\mathrm{Lip}(\Pi_\gamma)} \leqslant C(\beta, \gamma, \varepsilon)\|f\|_{\mathrm{Lip}(\Pi_\beta)}. \tag{2.1}$$

This lemma can be proved by the same method as Lemma 1.2. For semi-strips, additional efforts are required.

Lemma 2.2. *For any $0 < \beta < \gamma$, any $\varepsilon > 0$, and any analytic function $U\colon \Pi_\beta^+ \to \mathbb{C}$, there exists an analytic function $V\colon \Pi_\gamma^+ \to \mathbb{C}$ such that*

$$\|U - V\|_{L_\infty(\Pi_\beta^+ + 1)} \leqslant \varepsilon \quad \text{and} \quad \|V\|_{\mathrm{Lip}(\Pi_\gamma^+)} \leqslant C(\beta, \gamma, \varepsilon)\|U\|_{\mathrm{Lip}(\Pi_\beta)}. \tag{2.2}$$

We approximate U not on the whole semi-strip Π_β^+, but on a smaller set

$$\Pi_\beta^+ + 1 = \{z \in \mathbb{C} \mid |\Im z| \leqslant \beta,\ \Re z \geqslant 1\}.$$

Proof. We extend U to the whole strip Π_β preserving its Lipschitz constant in such a manner that the extension $\tilde{U}$ is constant when $\Re z \leqslant -\beta$. After that we convolve $\tilde{U}$ with χ_δ in the same manner as we did in the proof of Lemma 1.2 and get the function W. This function is not analytic on Π_β, however, its boundary values $f|_{\Im z = \pm\beta}$ allow analytic extensions to Π_γ. Denote these extensions by W_+ and W_- respectively. Consider the function $\tilde{W}\colon \Pi_\gamma \to \mathbb{C}$ given by the formula

$$\tilde{W} = \begin{cases} W_+, & \Im z \in [\beta, \gamma];\\ W, & z \in \Pi_\beta;\\ W_-, & \Im z \in [-\gamma, -\beta]. \end{cases}$$

As we have seen, $\tilde{W}$ is Lipschitz in Π_γ and approximates $\tilde{U}$ in Π_β if $\delta \leqslant K\varepsilon$ for sufficiently small constant K. The only problem is that $\tilde{W}$ is not analytic, namely,

$$\bar{\partial}\tilde{W} = \begin{cases} 0, & \Im z \in [\beta, \gamma];\\ \bar{\partial}\tilde{U} * \chi_\delta, & z \in \Pi_\beta;\\ 0, & \Im z \in [-\gamma, -\beta]. \end{cases} \tag{2.3}$$

Note that $\bar{\partial}\tilde{U}$ does not vanish on $[-\beta, 0] \times [-\beta, \beta]$ only and is bounded by $\|U\|_{\mathrm{Lip}}$ there. Therefore, $\bar{\partial}\tilde{W}$ is rapidly decaying at infinity,

$$|\bar{\partial}\tilde{W}|(z) \lesssim (1 + \delta^{-1}|\Re z|)^{-10}\|U\|_{\mathrm{Lip}}, \quad \text{when } \Re z > 0. \tag{2.4}$$

To make $\tilde{W}$ a smooth function, we convolve it with a non-negative C^∞-function of two variables supported in $[-\delta,\delta]^2$, having unit integral, and denote the result of such a convolution by $\tilde{\tilde{W}}$. Then,

$$\|\tilde{\tilde{W}} - \tilde{W}\|_{\Pi_\beta} \leqslant \varepsilon$$

and the inequality (2.4) holds for $\tilde{\tilde{W}}$ in the place of $\tilde{W}$ as well, provided δ is sufficiently small. What is more, $\tilde{\tilde{W}}$ is a smooth function, whose smoothness depends on δ. We consider the correction term

$$E(z) = \frac{1}{2\pi i}\int\limits_{\Pi_\gamma} \frac{\bar{\partial}\tilde{\tilde{W}}(\zeta)\,dm(\zeta)}{\zeta - z},$$

(we integrate with respect to the Lebesgue measure). Then, the function $V = \tilde{\tilde{W}} - E$ is analytic on Π_γ. We need to prove that E has small L_∞ norm on $\Pi_\beta^+ + 1$ and has bounded Lipschitz norm.

The first estimate:

$$\Big|\int\limits_{\Pi_\gamma} \frac{\bar{\partial}\tilde{\tilde{W}}(\zeta)\,dm(\zeta)}{z-\zeta}\Big| \overset{(2.4)}{\lesssim} \Big|\int\limits_{\Pi_\gamma} \frac{(1+\delta^{-1}|\Re\zeta|)^{-10}\,dm(\zeta)}{|z-\zeta|}\Big|\|U\|_{\mathrm{Lip}} \leqslant$$
$$\Big|\int\limits_{|\zeta-z|\leqslant\frac12} \frac{(1+\delta^{-1}|\Re\zeta|)^{-10}\,dm(\zeta)}{|z-\zeta|}\Big|\|U\|_{\mathrm{Lip}} + \Big|\int\limits_{\{|\zeta-z|\geqslant\frac12\}\cap\Pi_\gamma} \frac{(1+\delta^{-1}|\Re\zeta|)^{-10}\,dm(\zeta)}{|z-\zeta|}\Big|\|U\|_{\mathrm{Lip}} \leqslant$$
$$O(\delta^{10})\|U\|_{\mathrm{Lip}} + \Big(\int\limits_{\{|\zeta-z|\geqslant\frac12\}\cap\Pi_\gamma} \frac{dm(\zeta)}{|z-\zeta|^2}\Big)^{\frac12}\Big(\int\limits_{\{|\zeta-z|\geqslant\frac12\}\cap\Pi_\gamma} (1+\delta^{-1}|\Re\zeta|)^{-20}\,dm(\zeta)\Big)^{\frac12}\|U\|_{\mathrm{Lip}} = O\big(\sqrt{\delta}\big)\|U\|_{\mathrm{Lip}}$$

since $|z| \geqslant 1$. So, we may take $\varepsilon = \sqrt{\delta}\|U\|_{\mathrm{Lip}}$.

To control the Lipschitz norm of E, we simply use higher derivatives of $\tilde{\tilde{W}}$:

$$|\partial E(z)| = \Big|\frac{1}{2\pi i}\int\limits_{\Pi_\gamma} \frac{\bar{\partial}\tilde{\tilde{W}}(\zeta)\,dm(\zeta)}{(z-\zeta)^2}\Big| \lesssim \|\bar{\partial}\tilde{\tilde{W}}\|_{C^1} \lesssim_\delta \|\tilde{f}\|_{\mathrm{Lip}}.$$

□

References

[1] W. Smith, D. M. Stolyarov, A. Volberg, *Uniform approximation of Bloch functions and the boundedness of the integration operator on H^∞*, Adv. Math. **314** (2017), 185–202.