FRAMED MOTIVES OVER $\mathrm{Spec}\mathbb{Z}$.

Abstract. These are draft notes on the theory of framed motives over $\mathrm{Spec}\,\mathbb{Z}$. The theory of framed motives over a perfect fields is constructed by G. Garkusha and I. Panin and other authors [6], [7], [1], [9], [5], [4]. One of the main results is the computation of the Ω fibrant resolution of the motivic suspension spectra in the stable motivic homotopy category $\mathbf{SH}(k)$ be a perfect base field k in terms of the presheaves of framed correspondences. In the article we give such a formula in $\mathbf{SH}(\mathrm{Spec}\,Z)$.

The text is not finished and is dirty. It contains the statements, the constructions and the proof strategy, but some arguments are skipped. The text is uploaded currently as notes of the talk given at the conference "Algebraic groups, Motives and K-theory." at the Euler institute, St Petersburg 9-13 September 2019.

1. Introduction

The theory of framed motives was suggested by Voevodsky [15] and constructed by Garkusha and Panin in [6]. The theory gives the universal computational approach in the case of a prefect base field k to the stable motivic homotopy category $\mathbf{SH}(k)$ (constructed in [MV99], [11]). It allows to lift the basic computational techniques form theory of Voevodsky's motives $\mathbf{DM}(k)$ ([14], [12]) to $\mathbf{SH}(k)$. by using of so-called framed correspondences.

The theory (see [6]) provides the computations of the motivically fibrant Ω-resolutions of the suspension spectra of smooth schemes in $\mathbf{SH}(k)$ as follows. Let for a smooth scheme Y, $L_{\text{st-mot}}(Y)$ denote the Ω-resolutions of the suspension spectra of Y in the category $\mathbf{SH}(k)$ constructed by the stabilisation of the unstable motivic homotopy category $\mathbf{H}(k)$ with respect to $S^1 \wedge \mathbb{G}_m$ or with respect to $T = \mathbb{A}^1/(\mathbb{A}^1 - 0)$. Then there is the sequence of morphisms in the categories of T-spectra of presheaves on the category of smooth schemes Sm_k, $\mathrm{Spec}_T(\mathbf{Pre}(\mathrm{Sm}_k))$:

$$\Sigma^\infty_T X \to (((\Sigma^\infty_T X)^{\text{nis-fr}})^{\Delta_k})_{\text{nis}} \to L^T_{\text{st-mot}}(Y) \tag{1}$$

such that the first morphism in a stable motivic equivalences and the second morphism is a level wise sectionwise simplicial homotopy equivalence in positive degrees. In the above formulas the subscript $(-)_{\text{nis}}$ denotes the Nisnevich local fibrant resolution, the superscript $(-)^{\Delta_k}$ denotes the endo-functor $F \mapsto F(\Delta \times -)$ on the category of simplicial presheaves $\mathbf{Pre}(\mathrm{Sm}_k)$, and $X^{\text{nis-fr}}$ are the presheaves of the so called framed correspondences.

The presheaf $X^{\text{nis-fr}}$ of framed correspondences has a precise elementary (geometrical) description (see def. 1) on the one side, and in the same time by the fundamental Voevodsky's lemma there is an equivalence

$$X^{\text{nis-fr}} = \varinjlim_n Hom_{\mathbf{Sh}_{\text{nis}}}(- \wedge \mathbb{P}^n/\mathbb{P}^{n-1}, X \wedge T^n),$$

where $\mathrm{Hom}_{\mathbf{Sh}_{\text{nis}}}$ denotes the hom-set in the category of pointed Nisnevich sheaves on Sm_k.

In [5] the theory of framed motives is revisited with the ∞-categorical point of view. It appears that the reconstruction of $\mathbf{SH}(k)$ in terms of framed correspondences and framed motives can be given similarly to the ∞-categorical formulas

$$\mathbf{SH} \simeq \mathbf{Pre}(\mathrm{Sm}_S)[w_{\mathbb{A}^1}, w_{\text{nis}}][T^{-1}], \mathbf{DM} = \mathrm{Stab}(\mathbf{Pre}^{\text{Ab}}_\Sigma(\mathrm{Cor}_S)[w^{-1}_{\mathbb{A}^1}, w^{-1}_{\text{nis}}][\mathbb{G}^{-1}_m)$$

Namely there is an equivalence $\mathbf{SH}(k) \simeq \mathbf{SH}^{\text{fr}}(k)$ where

$$\mathbf{SH}^{\text{fr}} = \mathbf{Pre}(\mathbf{Corr}^{\text{fr}}_S)[w_{\mathbb{A}^1}, w_{\text{nis}}][T^{-1}]$$

and $\mathbf{Corr}^{\mathrm{fr}}_S$ is the ∞-category of framed correspondences. But in difference to case of $\mathbf{DM}(S)$ the category of framed correspondences $\mathbf{Corr}^{\mathrm{fr}}_S$ is essentially an ∞-category and not a classical one.

As shown in [10] by Marc Hoyois the categories $\mathbf{SH}^{\mathrm{fr}}$ satisfies the localisation property and consequently the equivalence

$$\mathbf{SH}(S) \simeq \mathbf{SH}^{\mathrm{fr}}(S)$$

holds for an arbitrary base scheme S. In the same times the formulas does not hold because of the un published Ayoub's counterexample to the $\mathbb{G}_m$-cancellation theorem [1].

The result of the article is

Theorem 1. *Let S be a scheme of a Krull dimension d. Consider the category of $(S^1, \mathbb{G}_m)$-bi-spectra (or T-spectra) of presheaves $\mathbf{Pre}^{S^1,\mathbb{G}_m}(S, \mathbb{Z}_c)$ (or $\mathbf{Pre}^T(S, \mathbb{Z}_c)$) with coefficients in the sheaf of rings $\mathbb{Z}_c$ that is (1) the constant sheaf of integers, i.e. $\mathbb{Z}_c = \mathbb{Z}$, if the residue fields of S are perfect; (2) the sheaf $\mathbb{Z}_{\mathrm{char}}$ that is the localisation of $\mathbb{Z}$ such that $l \in \mathbb{Z}^{\times}_{\mathrm{char}}(U) \Leftrightarrow l \notin \mathcal{O}^{\times}(U)$.*

Consider the stable motivic homotopy category with respect to Nisnevich topology over S:

$$\mathbf{SH}^{S^1,\mathbb{G}_m}_{\mathrm{nis},\mathbb{A}^1}(S, \mathbb{Z}_c) = \mathbf{Pre}^{S^1,\mathbb{G}_m}(S, \mathbb{Z}_c)[w^{-1}_{\mathbb{A}^1}, w^{-1}_{\mathrm{nis}}][(S^1 \wedge \mathbb{G}_m)^{-1}], \mathbf{SH}^{T}_{\mathrm{nis},\mathbb{A}^1}(S, \mathbb{Z}_c) = \mathbf{Pre}^{T}(S, \mathbb{Z}_c)[w^{-1}_{\mathbb{A}^1}, w^{-1}_{\mathrm{nis}}][(S^1 \wedge \mathbb{G}_m)^{-1}]$$

and denote by $L_{\text{st-mot}}$ the Ω-motivic resolution functor. The canonical morphisms

$$(2)\quad \begin{array}{rcl} L_{\text{st-mot}}((\Sigma^\infty_T Y)^{\mathrm{fr}}) & \leftarrow & L_{\mathrm{nis}}\Omega^\infty_{\mathbb{G}_m} L_{\mathbb{A}^1} L_{\varphi_d} L_{\mathbb{A}^1} L_{\varphi_{d-1}} L_{\mathbb{A}^1} \dots L_{\varphi_1} L_{\mathbb{A}^1}((\Sigma^\infty_{\mathbb{G}_m}\Sigma^\infty_T Y)^{\mathrm{fr}}) \\ & = & L_{\mathrm{nis}}(\varinjlim_l \Omega^l_{\mathbb{G}_m} L_{\mathbb{A}^1} L_{\varphi_d} L_{\mathbb{A}^1} L_{\varphi_{d-1}} L_{\mathbb{A}^1} \dots L_{\varphi_1} L_{\mathbb{A}^1}((\Sigma^l_{\mathbb{G}_m}\Sigma^\infty_T Y)^{\mathrm{fr}})) \\ L_{\text{st-mot}}((\Sigma^\infty_{S^1,\mathbb{G}_m} Y)^{\mathrm{fr}}) & \leftarrow & L_{\mathrm{nis}}\Omega^\infty_{\mathbb{G}_m} L_{\mathbb{A}^1} L_{\varphi_d} L_{\mathbb{A}^1} L_{\varphi_{d-1}} L_{\mathbb{A}^1} \dots L_{\varphi_1} L_{\mathbb{A}^1}(\Sigma^\infty_{\mathbb{G}_m}\Sigma^\infty_{S^1,\mathbb{G}_m} Y^{\mathrm{fr}}) \\ & = & L_{\mathrm{nis}}(\varinjlim_l \Omega^l_{\mathbb{G}_m} L_{\mathbb{A}^1} L_{\varphi_d} L_{\mathbb{A}^1} L_{\varphi_{d-1}} L_{\mathbb{A}^1} \dots L_{\varphi_1} L_{\mathbb{A}^1}((\Sigma^{\infty+l}_{\mathbb{G}_m}\Sigma^\infty_{S^1} Y)^{\mathrm{fr}})) \end{array}$$

is an equivalence in positive degrees in the category $\mathbf{Pre}^{S^1,\mathbb{G}_m}(S, \mathbb{Z}_c)$, where $\Sigma^\infty Y = \Sigma^\infty_{S^1,\mathbb{G}_m} Y$ or $\Sigma^\infty Y = \Sigma^\infty_T Y$, and see Def. 5 for the definition of the topologies φ_i.

1.1. **Conventions and notations.** All the categories of presheaves $\mathbf{Pre}$ on the categories of schemes are cansidered as ∞-categories of (simplicial) sets. In the superscript we write $\mathbf{Pre}^T$ for the T-spectra of presheaves, $\mathbf{Pre}^{S^1,\mathbb{G}_m}$ for the $(S^1, \mathbb{G}_m)$-bi-spectra of presheaves, and $\mathbf{Pre}^{\mathrm{fr}} = \mathbf{Pre}(\mathbf{Corr}^{\mathrm{fr}})$ denotes the framed presheaves, also we write $\mathrm{Sm}^{\mathrm{fr}} = \mathbf{Corr}^{\mathrm{fr}}$. In subscript we mean the localisations $\mathbf{Pre}_{\mathrm{nis}}$ is the subcategory of Nisnevich local objects, $\mathbf{Pre}_{\mathbb{A}^1}$ is the subcategory of $\mathbb{A}^1$-local objects, and $\mathbf{Pre}_\Sigma$ is the subcategory of radditive presheaves.

2. Recollection of framed correspondences.

In the section we recall the definitions of framed correspondences.

2.1. The graded category $\mathbf{Corr}^{\text{nis-fr}}_*$.

Definition 1. Let S be a scheme.

A level n Nisnevich framed correspondence form X to Y, $X, Y \in \mathrm{Sm}_S$ is given by the following data:

(i) a closed subscheme $Z \subset \mathbb{A}^n \times X$ finite over X.
(ii) a set of regular functions $\varphi_i \in \mathcal{O}((\mathbb{A}^n \times X)^h_Z)$, $i = 1, \dots n$, such that $Z(\varphi_1, \dots \varphi_n) = Z$.
(iii) a regular map $g\colon (\mathbb{A}^n \times X)^h_Z \to Y$

Denote by $\mathbf{Corr}^{\text{nis-fr}}_n(X, Y)$ the pointed set of the above data for a given X and Y pointed at the correspondence with $Z = \emptyset$. Set $\mathbf{Corr}^{\text{nis-fr}}_*(X, Y) = \bigvee_{n\geqslant 0} \mathbf{Corr}^{\text{nis-fr}}_n(X, Y)$.

A level n 1-th order framed correspondence form X to Y, $X, Y \in \mathrm{Sm}_S$ is given by the following data:

(i) a closed subscheme $Z \subset \mathbb{A}^n \times X$ finite over X.
(ii) a set of regular functions $\varphi_i \in \mathcal{O}((\mathbb{A}^n \times X)^{1\mathrm{th}}_h)$, $i = 1, \dots n$, such that $Z(\varphi_1, \dots \varphi_n) = Z$.
(iii) a regular map $g\colon (\mathbb{A}^n \times X)^{1\mathrm{th}}_h \to Y$.

Denote by $\mathbf{Corr}_n^{\text{1th-fr}}(X, Y)$ the pointed set of the above data for a given X and Y pointed at the element with $Z = \emptyset$. Set $\mathbf{Corr}_*^{\text{1th-fr}}(X, Y) = \bigvee_{n \geqslant 0} \mathbf{Corr}_n^{\text{1th-fr}}(X, Y)$

There is a natural way to define the graded composition of framed correspondences and make both of $\mathbf{Corr}_*^{\text{nis-fr}}$ and $\mathbf{Corr}_*^{\text{1th-fr}}$ being the graded categories.

Definition 2. Define the element $\sigma_X \in \mathbf{Corr}_1^{\text{nis-fr}}(X, X)$ as the correspondences given by ..

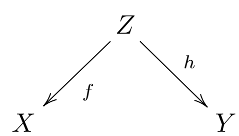
Define σ-suspension map $\mathbf{Corr}_n^{\text{nis-fr}}(X, Y) \to \mathbf{Corr}_{n+1}^{\text{nis-fr}}(X, Y)$, $\mathbf{Corr}_n^{\text{1th-fr}}(X, Y) \to \mathbf{Corr}_{n+1}^{\text{1th-fr}}(X, Y)$ as the map $c \mapsto \sigma_Y \circ c$.

Denote by $Y^{\text{nis-fr}}$ and $Y^{\text{1th-fr}}$ the presheaves on Sm_S with sections $Y^{\text{nis-fr}}(X) = \varinjlim_{n \geqslant 0} \mathbf{Corr}_n^{\text{nis-fr}}(X, Y)$, $Y^{\text{1th-fr}}(X) = \varinjlim_{n \geqslant 0} \mathbf{Corr}^{\text{1th-fr}}(X, Y)$ where the inductive limits are with respect to the σ-suspension maps.

2.2. The ∞-category $\mathbf{Corr}_S^{\text{tg-fr}}$.

Definition 3 (tangentially framed correspondences)**.** Let S be a scheme. A tangentially framed correspondence from X to Y, $X, Y \in \text{Sm}_S$ consists of the following data:

(i) A span



in Sch_S, where f is finite syntomic.

(ii) A trivialization $\tau\colon 0 \simeq L_f$ of the cotangent complex of f in the K-theory ∞-groupoid $\mathbf{K}(Z)$.

The ∞-category $\mathbf{Corr}_S^{\text{tg-fr}}(X, Y)$ is the ∞-groupoid of tangentially framed correspondences:

$$\mathbf{Corr}_S^{\text{tg-fr}}(X, Y) = \operatorname{colim}_{X \xleftarrow{f} Z \to Y} Maps_{K(Z)}((0, L_f)$$

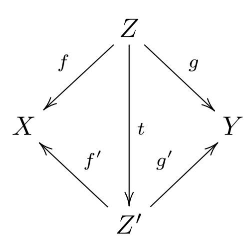
the colimit being taken over the groupoid of spans $X \xleftarrow{f} Z \to Y$ with f finite syntomic.

As shown in [5] there is an ∞-category $\mathbf{Corr}_S^{\text{tg-fr}}$ with a functor $\text{Sm}_S \to \mathbf{Corr}_S^{\text{tg-fr}}$ essentially isomorphic on objects, and such that the hom-spaces are equivalent to $\mathbf{Corr}_S^{\text{tg-fr}}(X, Y)$.

To simplify notations throw out the text we denote the category $\mathbf{Corr}_S^{\text{tg-fr}}$ by Sm_S^{fr}.

Explicitly, a 1-morphism $(Z, f, g, \tau) \to (Z', f', g', \tau')$ between tangentially framed correspondences consists of:

an isomorphism $t\colon Z \to Z'$ making the following diagram commute:



a path-homotopy $\beta\colon dt \circ \tau \to t^*(\tau')$ in $\mathbf{K}(Z)$, where $dt\colon Lf \simeq t^*(Lf')$ is the canonical equivalence.

By the functoriality of the cotangent complex, we obtain a functor $\mathbf{Corr}_S^{\text{tg-fr}}(-,-)\colon \text{Sch}_S^{\text{op}} \times \text{Sch}_S \to \text{Spc}$. We will denote by $Y^{\text{tg-fr}}$ the presheaf $X \mapsto \mathbf{Corr}_S^{\text{tg-fr}}(X, Y)$ on Sch.

2.3. **Smooth models.** The framed correspondences has the following motivically equivalent models, see [8], [5], [3].

Definition 4. Let S be a scheme.

A level n *1-th order framed correspondence* form X to Y, $X, Y \in \mathrm{Sm}_S$, is given by the following data:

(i) a closed subscheme $Z \subset \mathbb{A}^n \times X$ finite over X.
(ii) a set of regular functions $\varphi_i \in \mathcal{O}((\mathbb{A}^n \times X)^{1\mathrm{th}}_Z)$, $i = 1, \dots n$, such that $Z(f_1, \dots f_n) = Z$;
(iii) a regular map $g\colon (\mathbb{A}^n \times X)^{1\mathrm{th}}_Z \to Y$

Denote by $\mathbf{Corr}^{1\text{th-fr}}_n(X, Y)$ the set of the above data for a given X and Y pointed at the element with $Z = \emptyset$. Set $\mathbf{Corr}^{1\text{th-fr}}(X, Y) = \bigvee_{n \geqslant 0} \mathbf{Corr}^{1\text{th-fr}}_n(X, Y)$

A level n *normally framed correspondence* form X to Y, $X, Y \in \mathrm{Sm}_S$, is given by the following data:

(i) a closed subscheme $Z \subset \mathbb{A}^n \times X$ finite over X.
(ii) a set of regular functions $\varphi_i \in \mathcal{O}((\mathbb{A}^n \times X)^{1\mathrm{th}}_Z)$, $i = 1, \dots n$, such that $Z(\varphi_1, \dots \varphi_n) = Z$,
(iii) a regular map $g\colon Z \to Y$.

Denote by $\mathbf{Corr}^{\text{nr-fr}}_n(X, Y)$ the set of the above data for a given X and Y pointed at the element with $Z = \emptyset$. Set $\mathbf{Corr}^{\text{nr-fr}}(X, Y) = \bigvee_{n \geqslant 0} \mathbf{Corr}^{\text{nr-fr}}_n(X, Y)$. [1]

A level n *idempotent framed correspondence* form X to Y, $X, Y \in \mathrm{Sm}_S$, is given by the following data:

(i) a closed subscheme $Z \subset \mathbb{A}^n \times X$ finite over X.
(ii) a set of polynomials $\varphi_i \in \mathcal{O}(\mathbb{A}^n \times X)$, $i = 1, \dots n$, with leading term of f_i being t_i^d such that $Z(\varphi_1, \dots \varphi_n) = Z \amalg Z'$,
(iii) a regular map $g\colon Z \to Y$.

Denote by $\mathbf{Corr}^{\text{id-fr}}_{n,d}(X, Y)$ the set of the above data for a given X and Y pointed at the element with $Z = \emptyset$. Set $\mathbf{Corr}^{\text{id-fr}}(X, Y) = \bigvee_{n \geqslant 0} \mathbf{Corr}^{\text{id-fr}}_n(X, Y)$.

A level n *degree d idempotent framed correspondence* form X to Y, $X, Y \in \mathrm{Sm}_S$, is given by the following data:

(i) a set of polynomials $\varphi_i \in \mathcal{O}(\mathbb{A}^n \times X)$, $i = 1, \dots n$, with leading term of f_i being t_i^d such that $Z(\varphi_1, \dots \varphi_n) = Z \amalg Z'$,
(ii) a regular map $g\colon Z \to Y$.

Denote by $\mathbf{Corr}^{\text{id-fr}}_{n,d}(X, Y)$ the set of the above data for a given X and Y pointed at the element with $Z = \emptyset$. Set $\mathbf{Corr}^{\text{id-fr}}(X, Y) = \bigvee_{n \geqslant 0, n^3} \mathbf{Corr}^{1\text{th-fr}}_n(X, Y)$ with the morphisms in the inductive system defined as in [3].

Denote the presheaves of sets $Y^{1\text{th-fr}_n} = \mathbf{Corr}^{1\text{th-fr}}_n(-, Y)$, $Y^{\text{nr-fr}_n} = \mathbf{Corr}^{\text{nr-fr}}_n(-, Y)$, $Y^{\text{id-fr}_n} = \mathbf{Corr}^{\text{id-fr}}_n(-, Y)$, $Y^{\text{id-fr}_{n,d}} = \mathbf{Corr}^{\text{id-fr}}_{n,d}(-, Y)$.

All of this types of correspondences defines presheaves of pointed sets that are representable in the category of pointed smooth schemes. The first type of correspondences in addition defines a graded category $\mathbf{Corr}^{1\text{th-fr}}_*$ similarly to $\mathbf{Corr}^{\text{nis-fr}}_*$.

Proposition 1. *There are natural morphisms in the category* $\mathbf{Pre}_\Sigma(\mathrm{Sm}_S)$. $Y^{\text{nis-fr}} \to Y^{1\text{th-fr}} \to Y^{\text{nr-fr}} \leftarrow Y^{\text{id-fr}}$ *all of that induces trivial* $\mathbb{A}^1$*-fibrations in* $\mathbf{Pre}(\mathrm{AffSm}_S)$*, and consequently all the morphisms are motivic equivalences in* $\mathbf{Pre}_\Sigma(\mathrm{Sm}_S)$.

Proof. See [3]. □

Proposition 2. *For any smooth affine scheme Y the presheaves $Y^{1\text{th-fr}_n}$, $Y^{\text{nr-fr}_n}$ and $Y^{\text{id-fr}_n}$ are representable be smooth affine schemes.*

[1] The normally framed correspondences ware suggested in the specialist community in the unpublished notes by A. Neshitov, and redounded independently and studied in [5].

Proof. The cases of $Y^{\text{nr-fr}_n}$ and $Y^{\text{id-fr}_n}$ are covered by [5] and [3], the case of $Y^{\text{1th-fr}_n}$ is similar to the case of $Y^{\text{nr-fr}_n}$. □

Corollary 1. *For any smooth affine scheme Y, a scheme X and a closed subscheme $Z \subset X$, for any correspondence $c \in Y^{\text{nis-fr}}(Z)$ there is a lift to a correspondence in $T^{\text{nis-fr}}(X^h_Z)$.*

Proof. The claim follows from propositions 1, 2 by lemma 1. □

Lemma 1. *Let S be a scheme and $Z \subset S$ be a closed subscheme. Let Y be an (essentially) smooth affine scheme over S. Then for any S-scheme X any regular map $f\colon X \times_S Z \to Y$ passes throw a map $\widetilde{f}\colon X^h_Z \to Y$.*

Proof. Skipped. □

3. The subtopology ξ of the Nisnevich topology.

Definition 5. Let ξ be topology on Sch_S with coverings being etale morphisms $V \to U$ such that for any $x \in S$ there is a lift $U \times_S x \to V$.

The topology ξ is generated by the Nisnevich squares for the pairs of the form $(U, U \times_S Z)$ where U is an S-scheme and a closed subscheme in U that is equal to $U \times_S Z$ for an irreducible closed subscheme Z. The topology ξ is generated by the topologies φ_i for all integer i.

Define the topology φ_i on Sch_S as the topology generated by the Nisnevich squares for the pairs of the form $(U, U \times_S Z)$ where U is an S-scheme and a closed subscheme in U that is equal to $U \times_S Z$ for a closed subscheme Z of a pure codimension i.

Lemma 2. *For any scheme S of a Krull dimension d the topologies φ_i, $i \in 1, \dots d$, generates the topology ξ and the there is an equivalence of endofunctors $L_\xi \simeq L_{\varphi_d} L_{\varphi_{d-1}} \dots L_{\varphi_1}$ on the category $\mathbf{Pre}(\mathrm{Sm}_S)$, where L_ξ and L_{φ_i} denotes the localisation functor with respect to the local ξ or φ_i equivalences.*

Proof. Skipped. □

4. The reduction of (framed) Nisnevich motivic localisation to the ξ-motivic localisation.

Proposition 3. *Let S be a scheme and $x \in S$ be a point with the perfect residue field. Let V be an open subscheme $V \subset \mathbb{A}^1 \times S$ such that the complement $(\mathbb{A}^1 \times S) \setminus V$ is finite over the neighbourhood of x in S. Let F be a $\mathbb{A}^1$-local ξ-local radditive presheaf of abelian groups on the category $\mathrm{Sm}^{\mathrm{fr}}_S$. Then $L_{\mathrm{nis}}(F)(V) \simeq F(V)$.*

Proof. The claim follows form following lemma. □

Lemma 3. *Let S be a scheme and U be an essentially smooth local scheme over S. Let $\mathcal{C} \to \mathbb{A}^1_U$ be an etale morphism. Then for any point $x \in S$ with a perfect residue field we have the equalities*

$$H^i_{\mathrm{nis}}(\mathcal{C}^h_x, F) = 0, i > 0, H^0_{\mathrm{nis}}(\mathcal{C}^h_x, F) = F(\mathcal{C}^h_x)$$

for any homotopy invariant stable linear framed presheaf F.

Proof. The claim follows by the arguments similar to [7]. Alternatively, we can use the arguments of [7] to cover the base field case and the general case follows form the base field case because of the smooth representability results for the motivically equivalent versions of framed correspondences [5] or [3]. □

Corollary 2.

(1) The $\mathbb{A}^1$-local resolution endo-functor $L^{\xi,\mathrm{fr},Ab}_{\mathbb{A}^1}$ on the category $\mathbf{Sh}^{\mathrm{fr},\mathrm{Ab}}_\xi(S)$ is Nisnevich exact.

(2) The endo-functor $L^{\xi,\mathrm{fr},\mathrm{Ab}}_{\mathbb{G}_m}$ on the category $\mathbf{Sh}^{\mathrm{fr},\mathrm{Ab}}_{\xi,\mathbb{A}^1}(S)$ is Nisnevich exact.

(3) The endo-functor $L^{\xi,\mathrm{fr},\mathrm{Ab}}_{\mathbb{G}_m,\mathbb{A}^1}$ on the category $\mathbf{Sh}^{\mathrm{fr},\mathrm{Ab}}_\xi(S)$ is Nisnevich exact.

Proof. The point (1) follows form the proposition 3 applied to $V = \mathbb{A}^1 \times U$ for all $U \in Sm_S$. The proposition **??** applyied to $V = \mathbb{G}_m \times U$ for all $U \in Sm_S$ implies that the endofunctor $\Omega_{\mathbb{G}_m}$ on $\mathbf{Sh}^{\mathrm{fr,Ab}}_{\xi,\mathbb{A}^1}$ is Nisnevich exact. The endofunctor $\Sigma_{\mathbb{G}_m}$ preserves the Nisnevich squares and hence it is Nisnevich exact too. Thus $L_{\mathbb{G}_m} = \Omega^\infty_{\mathbb{G}_m}\Sigma^\infty_{\mathbb{G}_m}$ is Nisnevich exact , so point (2) is obtained. The point (3) follows form (2) and (1). □

Theorem 2 (See [6], and [5])**.** *Let S be a scheme. Then for any S-scheme Y the natural morphism $L_{\mathbb{A}^1}(Y^{\mathrm{fr}}) \amalg \dots L_{\mathbb{A}^1}(Y^{\mathrm{fr}}) \to L_{A^1}(Y^{\mathrm{fr}} \amalg \dots Y^{\mathrm{fr}})$ is an equivalence in $\mathbf{Pre}^{\mathrm{fr}}_\Sigma(S)$.*

Remark 1. The theorem 2 implies that the presheaves $L_{\mathbb{A}^1}(Y^{\mathrm{fr}})$ are equipped with the structure of E_∞ spaces or ∞-commutative monoids.

Lemma 4. *The functor $\mathbf{Pre}_\Sigma \to \mathbf{Pre}^{\mathrm{fr}}_\Sigma$ over a base schemes S is Nisnevich and ξ exact.*

Theorem 3. *Let S be a scheme. Consider the category of $(S^1,\mathbb{G}_m)$-bi-spectra of framed radditive presheaves $\mathbf{Pre}^{S^1,\mathbb{G}_m,\mathrm{fr}}_\Sigma(S,\mathbb{Z}_c)$ with coefficients in the sheaf of rings $\mathbb{Z}_c$ that is (1) the constant sheaf of integers, i.e. $\mathbb{Z}_c = \mathbb{Z}$, if the residue fields of S are perfect; (2) the sheaf $\mathbb{Z}_{\mathrm{char}}$ that is the localisation of $\mathbb{Z}$ such that $l \in \mathbb{Z}^\times_{\mathrm{char}}(U) \Leftrightarrow l \notin \mathcal{O}^\times(U)$.*

Consider the stable framed motivic homotopy category of $(S^1,\mathbb{G}_m)$-bispectra over S with the $\mathbb{Z}_c$ coefficients:

$$\mathbf{SH}^{\mathrm{fr}}(S,\mathbb{Z}_c) \simeq \mathbf{Pre}^{S^1,\mathbb{G}_m,\mathrm{fr}}_\Sigma(S,\mathbb{Z}_c)[w^{-1}_{\mathbb{A}^1}, w_{\mathrm{nis}}][(S^1\wedge\mathbb{G}_m)^{-1}]$$

Then the natural morphism

$$L_{\text{st-mot}}(\Sigma^\infty_{S^1,\mathbb{G}_m}Y) \leftarrow L_{\mathrm{nis}}(L_{\xi,\mathbb{A}^1,S^1,\mathbb{G}_m}((\Sigma^\infty_{S^1,\mathbb{G}_m}Y)^{\mathrm{fr}}))$$

is an equivalence in positive degrees.

Proof. It follows by theorem 2 that

$$L^{\Sigma,\mathrm{fr}}_{\mathbb{A}^1,S^1}(\Sigma^\infty_{S^1,\mathbb{G}_m}Y)^{\mathrm{fr}}) \simeq L_{\mathbb{A}^1}(\Sigma^\infty_{S^1,\mathbb{G}_m}Y)^{\mathrm{fr}})^{\mathrm{gp}}. \tag{3}$$

it follows by the corollary 2.(3) that the functor $L^{\xi,\mathrm{fr}}_{\mathbb{A}^1,\mathbb{G}_m}$ on $\mathbf{Sh}^{\mathrm{fr,gpCMon}}_\xi$ is Nisnevich exact, and hence there is an equivalence of endo-functors

$$L^{\xi,\mathrm{fr,gpCMon}}_{\mathbb{G}_m,\mathbb{A}^1,\mathrm{nis}} \simeq L^{\xi,\mathrm{fr,gpCMon}}_{\mathrm{nis}} L^{\xi,\mathrm{fr,gpCMon}}_{\mathbb{G}_m,\mathbb{A}^1} \tag{4}$$

on the category $\mathbf{Sh}^{\mathrm{fr,gpCMon}}_\xi(S)$.

Now the claim follows form (3) and (4). Namely, by (3) $(L_{\mathbb{A}^1}(\Sigma^\infty_{S^1,\mathbb{G}_m}Y)^{\mathrm{fr}})^{\mathrm{gp}}$ is a group like commutative monoid, so $L^{\mathrm{fr}}_{\mathbb{A}^1,S^1}((\Sigma^\infty_{S^1,\mathbb{G}_m}Y)^{\mathrm{fr}}) \simeq (L_{\mathbb{A}^1}(\Sigma^\infty_{S^1,\mathbb{G}_m}Y)^{\mathrm{fr}})^{\mathrm{gp}}$. Then by (4)

$$L^{\mathrm{fr}}_{smot}(L^{\mathrm{fr}}_{\mathbb{A}^1,S^1}(\Sigma^\infty_{S^1,\mathbb{G}_m}Y)^{\mathrm{fr}}) \simeq L^{\mathrm{fr}}_{\mathrm{nis}}L^{\mathrm{fr}}_{\xi,\mathbb{G}_m,\mathbb{A}^1}(L^{\mathrm{fr}}_{\mathbb{A}^1,S^1}(\Sigma^\infty_{S^1,\mathbb{G}_m}Y)^{\mathrm{fr}}).$$

Thus

$$\begin{aligned} L^{\mathrm{fr}}_{smot}((\Sigma^\infty_{S^1,\mathbb{G}_m}Y)^{\mathrm{fr}}) \simeq L^{\mathrm{fr}}_{smot}(L^{\mathrm{fr}}_{\mathbb{A}^1,S^1}(\Sigma^\infty_{S^1,\mathbb{G}_m}Y)^{\mathrm{fr}}) \simeq L^{\mathrm{fr}}_{\mathrm{nis}}L^{\mathrm{fr}}_{\xi,\mathbb{G}_m,\mathbb{A}^1}(L^{\mathrm{fr}}_{\mathbb{A}^1,S^1}(\Sigma^\infty_{S^1,\mathbb{G}_m}Y)^{\mathrm{fr}}) \simeq \\ L^{\mathrm{fr}}_{\mathrm{nis}}L^{\mathrm{fr}}_{\xi,\mathbb{G}_m,\mathbb{A}^1}(L_{\mathbb{A}^1}(\Sigma^\infty_{S^1,\mathbb{G}_m}Y)^{\mathrm{fr}})^{\mathrm{gp}}) \simeq L^{\mathrm{fr}}_{\mathrm{nis}}(L^{\mathrm{fr}}_{\xi,\mathbb{G}_m,\mathbb{A}^1}(\Sigma^\infty_{S^1,\mathbb{G}_m}Y)^{\mathrm{fr}})^{\mathrm{gp}} \end{aligned}$$

□

5. The stable ξ-motivic localisation

Theorem 4. *Let S be a scheme of a Krull dimension d. Consider the category of bi-spectra of framed radditive presheaves $\mathbf{Pre}^{S^1,\mathbb{G}_m,\mathrm{fr}}_\Sigma(S)$, and the stable motivic homotopy category with respect to ξ-topology over S:*

$$\mathbf{SH}^{S^1,\mathbb{G}_m,\mathrm{fr}}_{\Sigma,\xi,\mathbb{A}^1}(S) = \mathbf{Pre}^{S^1,\mathbb{G}_m,\mathrm{fr}}_\Sigma(S)[w^{-1}_{\mathbb{A}^1}, w^{-1}_\xi][(S^1\wedge\mathbb{G}_m)^{-1}],$$

and denote by $L_{\text{st-mot}}$ *the* Ω*-motivic resolution functor. The canonical morphisms*

$$L_{\text{st-mot}}((\Sigma^{\infty}_{S^1,\mathbb{G}_m} Y)^{\text{fr}}) \leftarrow \Omega^{\infty}_{\mathbb{G}_m} L_{\mathbb{A}^1} L_{\varphi_d} L_{\mathbb{A}^1} L_{\varphi_{d-1}} L_{\mathbb{A}^1} \dots L_{\varphi_1} L_{\mathbb{A}^1}((\Sigma^{\infty}_{\mathbb{G}_m}\Sigma^{\infty}_{S^1,\mathbb{G}_m} Y)^{\text{fr}})$$

are an equivalences in $\mathbf{Pre}^{S^1,\mathbb{G}_m,\text{fr}}_{\Sigma}(S)$ *in positive degrees in the category* $\mathbf{Pre}^{S^1,\mathbb{G}_m,\text{fr}}_{\Sigma}(S,\mathbb{Z}_c)$*.*

Proof. The morphism is clearly a motivic equivalence. So we need to prove that it is Ω_{S^1}, $\Omega_{\mathbb{G}_m}$, $\mathbb{A}^1$-local and ξ-local.

(Ω_{S^1}-spectrum) The claim follows since $L_{\mathbb{A}^1}(\Sigma^{\infty}_{S^1,\mathbb{G}_m} Y)^{\text{fr}}$ is a commutative monoid, and the functors L_{φ_i} and $L_{\mathbb{A}^1}$ preserves the commutative monoids.

($\Omega_{\mathbb{G}_m}$-spectrum) The claim follows because of that the external functor is $\Omega^{\infty}_{\mathbb{G}_m}$

($\mathbb{A}^1$-local) The claim follows, since $\Omega^{\infty}_{\mathbb{G}_m} L_{\mathbb{A}^1} \simeq L_{\mathbb{A}^1}\Omega^{\infty}_{\mathbb{G}_m}$.

(ξ-local) We prove by induction on i that for any S the morphisms

$$\begin{array}{l} L_{\text{st-mot}}((\Sigma^{\infty}_{S^1,\mathbb{G}_m} Y)^{\text{fr}}) \leftarrow \Omega^{\infty}_{\mathbb{G}_m} L_{\mathbb{A}^1} L_{\varphi_d} L_{\mathbb{A}^1} L_{\varphi_{d-1}} L_{\mathbb{A}^1} \dots L_{\varphi_1}(L_{A^1}(\Sigma^{\infty}_{\mathbb{G}_m}\Sigma^{\infty}_{S^1,\mathbb{G}_m} Y^{\text{fr}}))^{\text{gp}} \\ L_{\text{mot}}((\Sigma^{\infty}_{S^1,\mathbb{G}_m} Y)^{\text{fr}}) \leftarrow L_{\mathbb{A}^1} L_{\varphi_d} L_{\mathbb{A}^1} L_{\varphi_{d-1}} L_{\mathbb{A}^1} \dots L_{\varphi_1}(L_{A^1}(\Sigma^{\infty}_{\mathbb{G}_m}\Sigma^{\infty}_{S^1,\mathbb{G}_m} Y^{\text{fr}}))^{\text{gp}} \end{array}$$

is an equivalence over $S-\overline{S^{(i)}}$. Consider the base change of the above morphisms over $S-\overline{S^{(i)}}$. The following sequence

$$\begin{array}{r} L_{\varphi_i}\Omega^{\infty}_{\mathbb{G}_m} L_{A^1} L_{\varphi_d} \dots L_{\varphi_1} L_{\mathbb{A}^1} \to \Omega^{\infty}_{\mathbb{G}_m} L_{\varphi_i} L_{A^1} L_{\varphi_d} \dots L_{\varphi_1} L_{\mathbb{A}^1} \simeq \\ \Omega^{\infty}_{\mathbb{G}_m} L_{\cup_i} L_{\varphi_d} L_{\mathbb{A}^1} \dots L_{\varphi_1} L_{\mathbb{A}^1} \simeq \Omega^{\infty}_{\mathbb{G}_m} L_{\cup_i} L_{\mathbb{A}^1} \dots L_{\varphi_1} L_{\mathbb{A}^1} \to \\ \Omega^{\infty}_{\mathbb{G}_m} L_{\mathbb{A}^1} L_{\cup_i} L_{\varphi_{d-1}} \dots L_{\varphi_1} L_{\mathbb{A}^1} \to \dots \to \Omega^{\infty}_{\mathbb{G}_m} L_{\mathbb{A}^1} L_{\cup_i} L_{\varphi_{i-1}} \dots L_{\varphi_1} L_{\mathbb{A}^1} \to \\ \Omega^{\infty}_{\mathbb{G}_m} L_{\mathbb{A}^1} L_{\varphi_i} L_{\mathbb{A}^1} L_{\varphi_{i-1}} \dots L_{\varphi_1} L_{\mathbb{A}^1} \to \Omega^{\infty}_{\mathbb{G}_m} L_{\mathbb{A}^1} L_{\varphi_d} L_{\mathbb{A}^1} \dots L_{\varphi_1} L_{\mathbb{A}^1} \end{array}$$

gives the inverse morphism. □

Theorem 5. *Let* S *be a scheme of a Krull dimension* d*. Consider the category of bi-spectra of radditive presheaves* $\mathbf{Pre}^{S^1,\mathbb{G}_m}_{\Sigma}(S)$*, and the stable motivic homotopy category with respect to* ξ*-topology over* S*:*

$$\mathbf{H}^{S^1,\mathbb{G}_m}_{\Sigma,\xi,\mathbb{A}^1}(S) = \mathbf{Pre}_{\Sigma}(S)[w^{-1}_{\mathbb{A}^1}, w^{-1}_{\xi}], \mathbf{SH}^{S^1,\mathbb{G}_m}_{\Sigma,\xi,\mathbb{A}^1}(S) = \mathbf{Pre}^{S^1,\mathbb{G}_m}_{\Sigma}(S)[w^{-1}_{\mathbb{A}^1}, w^{-1}_{\xi}][(S^1\wedge\mathbb{G}_m)^{-1}],$$

and denote by $L_{\text{st-mot}}$ *the* Ω*-motivic resolution functor. The canonical morphisms*

$$\begin{array}{lcl} L_{\text{st-mot}}(\Sigma^{\infty}_{S^1,\mathbb{G}_m} Y) & \leftarrow & \Omega^{\infty}_{\mathbb{G}_m} L_{\mathbb{A}^1} L_{\varphi_d} L_{\mathbb{A}^1} L_{\varphi_{d-1}} L_{\mathbb{A}^1} \dots L_{\varphi_1} L_{A^1}(\Sigma^{\infty}_{\mathbb{G}_m}\Omega_{S^1}\Sigma^{\infty}_{S^1,\mathbb{G}_m} Y), \\ L_{\text{mot}}(Y) & \leftarrow & L_{\mathbb{A}^1} L_{\varphi_d} L_{\mathbb{A}^1} L_{\varphi_{d-1}} L_{\mathbb{A}^1} \dots L_{\varphi_1} L_{A^1}(Y) \end{array}$$

are an equivalences in $\mathbf{Pre}^{S^1,\mathbb{G}_m,\text{fr}}_{\Sigma}(S)$ *in positive degrees in the category* $\mathbf{Pre}^{S^1,\mathbb{G}_m}_{\Sigma}(S,\mathbb{Z}_c)$*.*

Remark 2. We can get even stronger formula

$$L_{\text{mot}}(Y) \leftarrow L_{\mathbb{A}^1} L_{\varphi'_d} L_{\mathbb{A}^1} L_{\varphi'_{d-1}} L_{\mathbb{A}^1} \dots L_{\varphi'_1} L_{\mathbb{A}^1}(\Sigma^{\infty}_{\mathbb{G}_m}\Sigma^{\infty}_{S^1,\mathbb{G}_m} Y)$$

and similarly with other formulas. Here φ' denotes the topologies on the subcategory of essentially smooth schemes $\text{EssSm}^{\xi}_S \subset \text{EssSm}_S$ spanned by the objects of the form $U^h_x \times V$, $U, V \in \text{Sm}_S$. The coverings of U a essentially smooth schemes of φ'_i are the same as for φ_i if for any $x \in S, \operatorname{codim}_S x > i$ $U \times_S x = \emptyset$, and the covering are trivial otherwise. So in the above formula we use the adjunction between $\mathbf{Pre}(Sm_S) \leftrightarrow \mathbf{Pre}(\text{EssSm}^{\xi}_S)$ that leads to an equivalence on the subcategories of ξ-local objects.

The advantage of such presentation is that the intersection of topologies φ'_I and φ'_J is equal to $\varphi'_{I\cap J}$ and $I, J \subset \{1, \dots d\}$, and φ_I is generated by $\varphi_i, \imath \in I$ and similarly for φ'_J.

6. The stable Nisnevich motivic localisation

Theorem 6. *Let S be a scheme of a Krull dimension d. Consider the category of $(S^1,\mathbb{G}_m)$-bi-spectra (or T-spectra) of presheaves $\mathbf{Pre}^{S^1,\mathbb{G}_m}_{\Sigma}(S,\mathbb{Z}_c)$ (or $\mathbf{Pre}^{T}_{\Sigma}(S,\mathbb{Z}_c)$) with coefficients in the sheaf of rings $\mathbb{Z}_c$ that is (1) the constant sheaf of integers, i.e. $\mathbb{Z}_c=\mathbb{Z}$, if the residue fields of S are perfect; (2) the sheaf $\mathbb{Z}_{\mathrm{char}}$ that is the localisation of $\mathbb{Z}$ such that $l\in\mathbb{Z}^{\times}_{\mathrm{char}}(U)\Leftrightarrow l\notin\mathcal{O}^{\times}(U)$.*

Consider the stable motivic homotopy category with respect to Nisnevich topology over S:

$$\mathbf{SH}^{S^1,\mathbb{G}_m}_{\mathrm{nis},\mathbb{A}^1}(S,\mathbb{Z}_c)=\mathbf{Pre}^{S^1,\mathbb{G}_m}(S,\mathbb{Z}_c)[w^{-1}_{\mathbb{A}^1},w^{-1}_{\mathrm{nis}}][(S^1\wedge\mathbb{G}_m)^{-1}],\mathbf{SH}^{T}_{\mathrm{nis},\mathbb{A}^1}(S,\mathbb{Z}_c)=\mathbf{Pre}^{T}(S,\mathbb{Z}_c)[w^{-1}_{\mathbb{A}^1},w^{-1}_{\mathrm{nis}}][(S^1\wedge\mathbb{G}_m)^{-1}]$$

and denote by $L_{\text{st-mot}}$ the Ω-motivic resolution functor. The canonical morphisms

$$L_{\text{st-mot}}((\Sigma^{\infty}_{S^1,\mathbb{G}_m}Y)^{\mathrm{fr}})\leftarrow L_{\mathrm{nis}}\Omega^{\infty}_{\mathbb{G}_m}L_{\mathbb{A}^1}L_{\varphi_d}L_{\mathbb{A}^1}L_{\varphi_{d-1}}L_{\mathbb{A}^1}\dots L_{\varphi_1}L_{A^1}((\Sigma^{\infty}_{\mathbb{G}_m}\Sigma^{\infty}_{S^1,\mathbb{G}_m}Y)^{\mathrm{fr}}),\tag{5}$$

$$L_{\text{st-mot}}((\Sigma^{\infty}_{T}Y)^{\mathrm{fr}})\leftarrow L_{\mathrm{nis}}\Omega^{\infty}_{\mathbb{G}_m}L_{\mathbb{A}^1}L_{\varphi_d}L_{\mathbb{A}^1}L_{\varphi_{d-1}}L_{\mathbb{A}^1}\dots L_{\varphi_1}L_{A^1}((\Sigma^{\infty}_{\mathbb{G}_m}\Sigma^{\infty}_{T}Y)^{\mathrm{fr}}),\tag{6}$$

is an equivalence in positive degrees in the categories $\mathbf{Pre}^{S^1,\mathbb{G}_m}(S,\mathbb{Z}_c)$,$\mathbf{Pre}^{T}(S,\mathbb{Z}_c)$, where $\Sigma^\infty Y=\Sigma^{\infty}_{S^1,\mathbb{G}_m}Y$ or $\Sigma^\infty Y=\Sigma^\infty_T Y$.

Proof. Firstly consider the case of a scheme S with a perfect residue fields. The case of $(S^1,\mathbb{G}_m)$-bi-spectra follows form the theorem 3 and theorem 4 with using of the equivalence of the categories $\mathbf{SH}(S)\simeq\mathbf{SH}^{\mathrm{fr}}(S)$ [10]. The case of T-spectra follows form the case of $(S^1,\mathbb{G}_m)$-bi-spectra with using of the Cone theorem, see proposition 5.

Now the case of a general S follows by the following (standard) argument. For any local scheme S of characteristic p there is a pro-finite flat morphism $\widetilde{S}\to S$ given by the sequence of finite flat morphisms of the form $pr\colon\operatorname{Spec}(\mathcal{O}(X)[t]/(t^p-a))\to\operatorname{Spec}\mathcal{O}(X)$, and such that $\widetilde{S}$ has the perfect residue filed. Now since $pr\circ\langle t^p-a\rangle\overset{\mathbb{A}}{\sim}p_\epsilon$ in $\mathrm{Sm}^{\mathrm{fr}}$, where $\langle t^p-a\rangle\colon\operatorname{Spec}\mathcal{O}(S)\to\operatorname{Spec}(\mathcal{O}(S)[t]/(t^p-a))$ is a framed correspondence with the framing given by the polynomial t^p-a, the claimed equivalence holds in the category $\mathbf{SH}(S,\mathrm{GW}_{\mathrm{char}})$, where $\mathrm{GW}_{\mathrm{char}}$ is the localisation of the ring GW obtained by inverting in $\mathrm{GW}(U)$, $U\subset S$, the elements p_ϵ, for $p=\mathrm{char}\mathcal{O}(x)$, where $x\in U$ is an point. Now the claim follows by lemma 10. □

Corollary 3.
(1) Let S be a scheme of a Krull dimension d. For any $Y\in Sm_S$, the Nisnevich sheaves $\underline{\varphi}^{i+j,j}_s(Y)$ on Sm_S associated with the presheaves $[-,Y\wedge\mathbb{G}_m^j\wedge S^i]_{\mathbf{SH}(S)}$ are trivial for all $i>d=\dim S$.
(2) For any local S of the Krull dimension d the group $[S,V\wedge\mathbb{G}_m^d\wedge S^d]_{\mathbf{SH}(S)}$ is non-trivial, where $V\subset\mathbb{G}_m^d\times S$, $V=\mathbb{G}_m^d\times S-Z(f_1\cdot f_2cdotsf_d)-Z((f_1t_1-1)(f_2t_2-1)\cdots(f_dt_d-1))$, t_i denotes the coordinate functions on $\mathbb{G}_m^d$, and $f_1,\dots f_d\in\mathcal{O}(S)$ be a functions such that we have the strongly increasing sequence of vanishing locus $Z(f_1,\dots f_d)\neq Z(f_1,\dots f_{d-1})\neq\dots Z(f_1)$.

Proof. 1) In view of the theorem the vanishing of the sheaf $[-,Y\wedge\mathbb{G}_m^j\wedge S^i]_{\mathbf{SH}(S)}$ follows from the vanishing of the presheaf $[-,Y\wedge\mathbb{G}_m^j\wedge S^i]_{\mathbf{SH}_\xi(S)}$. To prove the last statement we consider the germs of the presheaf over the pro-schemes $S-\overline{S^c}$. We prove by induction on $c\in\mathbb{Z}_{>0}$ that $[U\times_S(S-\overline{S^c}),Y\wedge\mathbb{G}_m^j\wedge S^i]_{\mathbf{SH}(S)}=0$ for all $i>c$ and local U. The base of induction is $c=1$ and the claim follows since the presheaves $[-\wedge\mathbb{G}_m^l,L_{\varphi_1}L_{\mathbb{A}^1}(\Sigma^l_{\mathbb{G}_m}\Sigma^\infty_T Y)$ vanishes for all Y, since $\forall Y\in Sm_S$ the cohomologies $H^i_{\varphi_1}(\mathbb{G}_m\wedge-,L^{\mathrm{fr,Ab}}_{\mathbb{A}^1}(Y^{\mathrm{fr,Ab}})$ vanishes for $i>0$, where $Y^{\mathrm{fr,Ab}}$ is the linearisation of Y^{fr}. The induction step is similar.

2) To prove that the sheaves associated with the presheaves $[-,V\wedge\mathbb{G}_m^j\wedge S^i]_{\mathbf{SH}(S)}$ are non-trivial we prove by induction no $c\in\mathbb{Z}>0$ that the germs of presheaves $[-,V]_{\mathbf{SH}_\xi(S)}$ on $S-\overline{S^{(c)}}$ are non-torsion. The base case for $c=1$ follows since the image of the identity map $\mathbb{G}_m^l\wedge V\to\mathbb{G}_m^l\wedge V$ along the map $H^0(\mathbb{G}_m^l\wedge V,L^{\mathrm{fr,Ab}}_{\mathbb{A}^1}((\mathbb{G}_m^l\wedge V)^{\mathrm{fr}}))\to H^1_{\varphi_1}(\mathbb{G}_m^{\wedge l}\wedge\mathbb{G}_m\wedge S,L^{\mathrm{fr,Ab}}_{\mathbb{A}^1}((\mathbb{G}_m^{\wedge l}\wedge V)^{\mathrm{fr}}))$ is non-torsion. □

Corollary 4. *Let S Let $Y\in Sm_S$ and $U\in\mathrm{Sm}_S$. Then $[U,Y\wedge S^i\wedge\mathbb{G}_m^j]=0$ for all $i>\dim U$.*

Proof. The case of $U = S$ follows by the formula from the theorem with the induction similar to the previous corollary, the general case follows by the base change argument. □

Let us point that the connectivity theorem in the category $\mathbf{SH}_{S^1}(S)$ over the noetherian domain S with infinite residue fields is proven in [13] and [2]. In the case of the category $\mathbf{SH}(S)$ the connectivity follows in the case of an arbitrary base scheme S by the argument of [4], [5].

7. Appendix 1: Correspondences of open pairs. The Cone theorem.

In the appendix we give the proof of the motivic equivalence $\langle Y/U\rangle^{\mathrm{fr}} \simeq Y^{\mathrm{fr}}/U^{\mathrm{fr}}$ for a smooth scheme $Y \in \mathrm{Sm}_S$ over a base scheme S and an open subscheme $U \subset Y$.

Definition 6. A level n Nisnevich framed correspondence form (X, U) to (Y, V), where $X, Y \in \mathrm{Sm}_S$, $U \subset X$ and $V \subset Y$ are open, is given by the following data:

(i) a closed subscheme $Z \subset \mathbb{A}^n \times X$ finite over X, and such that $Z \times_X U = \emptyset$.
(ii) a set of regular functions $\phi_i \in \mathcal{O}((\mathbb{A}^n \times X)^h_Z)$, $i = 1, \dots n$, and a regular map $g\colon (\mathbb{A}^n \times X)^h_Z \to Y$ such that $(\mathbb{A}^n \times X)^h_Z \times_{\phi_1,\dots\phi_n,g} (0 \times (Y \setminus V)) = Z$.

A level n *lower normally framed correspondence* form $\langle X, U\rangle$ to $\langle Y, V\rangle$, where $X, Y \in \mathrm{Sm}_S$, $U \subset X$ and $V \subset Y$ are open, is given by the following data:

(i) a closed subschemes $Z \subset W \subset \mathbb{A}^n \times X$ finite over X, such that $U \times_X Z = \emptyset$.
(ii) a set of regular functions $\phi_i \in \mathcal{O}((\mathbb{A}^n \times X)^{1\mathrm{th}}_Z)$, $i = 1, \dots n$, such that $Z(\phi_1, \dots \phi_n) = W$,
(iii) a regular map $g\colon W \to Y$ such that $W \times_Y (Y \setminus V) = Z$.

A level n *quasi-finite normally framed correspondence* form a pair $\langle X, U\rangle$ to $\langle Y, V\rangle$, where $X, Y \in \mathrm{Sm}_S$, $U \subset X$ and $V \subset Y$ are open, is given by the following data:

(i) a closed subscheme $W \subset \mathbb{A}^n \times X$ quasi-finite over X;
(ii) a set of regular functions $\phi_i \in \mathcal{O}((\mathbb{A}^n \times X)^{1\mathrm{th}}_W)$, $i = 1, \dots n$, such that $Z(\phi_1, \dots \phi_n) = W$;
(iii) a regular map $g\colon W \to Y$ such that the subscheme $Z \overset{\mathrm{def}}{=} W \times_Y (Y \setminus V)$ is finite over X and $U \times_X Z = \emptyset$.

A *quasi-finite tangentially framed correspondence* is defined by the data

(i) A span $X \overset{f}{\leftarrow} W \overset{g}{\to} Y$ in SchPair with f quasi-finite syntomic, and such that the subscheme $Z \overset{\mathrm{def}}{=} W \times_Y (Y \setminus V)$ is finite over X and $U \times_X Z = \emptyset$.
(ii) path in K-theory on W

The schemes Z in the above definitions are called as a support of a correspondence, and the scheme W (if it is) is called as a big-support.

Proposition 4. *The morphism $\langle Y/V\rangle^{\mathrm{qfnr\text{-}fr}} \to \langle Y/V\rangle^{\mathrm{qftg\text{-}fr}}$ induces the equivalence in $\mathbf{Pre}_{\mathrm{nis},\mathbb{A}^1}$.*

Proof. Skipped. Similar to [5]. □

Proposition 5. *The morphism*

$$\langle Y/V\rangle^{\mathrm{nis\text{-}fr}} \to \langle Y/V\rangle^{\mathrm{lownr\text{-}fr}} \leftarrow \langle Y/V\rangle^{\mathrm{qfnr\text{-}fr}}$$

induces equivalences in $\mathbf{Pre}_{\mathrm{nis},\mathbb{A}^1}(S)$. Moreover the morphisms above restricted to the category AffSm *are the trivial $\mathbb{A}$-fibrations.*

Proof. The first claim follows form the second, and by lemma 5 to prove the second claim it is enough to prove the lifting property of the above morphisms with respect to closed immersions. Let X be an affine scheme, and $X_0 \subset X$ be a closed subscheme.

Consider the case of the morphism $\langle Y/V\rangle^{\mathrm{nis\text{-}fr}} \to \langle Y/V\rangle^{\mathrm{lownr\text{-}fr}}$. Let

$$(Z, W, \phi, g) \in \mathbf{Corr}^{\mathrm{lownr\text{-}fr}}(X, Y/V), (Z_0, \tilde{\phi}_0, \tilde{g}_0) \in \mathbf{Corr}^{\mathrm{nis\text{-}fr}}(X_0, Y/V),$$

define an element in $\mathbf{Corr}^{\text{lownr-fr}}(X,Y/V)\times_{\mathbf{Corr}^{\text{lownr-fr}}(X_0,Y/V)}\mathbf{Corr}^{\text{nis-fr}}(X_0,Y/V)$. Since Y is smooth and consequently $Y\times\mathbb{A}^n$ is smooth, it follows by lemma 1 that there is a regular map $(\tilde{\phi},\tilde{g})\colon(\mathbb{A}^n\times X)^h_Z\to\mathbb{A}^n\times Y$, such that

$$\tilde{\phi}\big|_{(\mathbb{A}^n\times X_0)^h_{Z_0}}=\tilde{\phi}_0\big|_{(\mathbb{A}^n\times X_0)^h_{Z_0}},\quad \tilde{\phi}\big|_{(\mathbb{A}^n\times X)^{1\text{th}}_Z}=\phi\big|_{(\mathbb{A}^n\times X)^{1\text{th}}_Z},$$
$$\tilde{g}\big|_{(\mathbb{A}^n\times X_0)^h_{Z_0}}=\tilde{g}_0\big|_{(\mathbb{A}^n\times X_0)^h_{Z_0}},\quad \tilde{g}\big|_W=g.$$

Define now the required lift as the correspondence

$$(Z,\tilde{\phi},\tilde{g})\in\mathbf{Corr}^{\text{nis-fr}}(X,Y/V).$$

Consider the case of the morphism $\langle Y/V\rangle^{\text{qfnr-fr}}\to\langle Y/V\rangle^{\text{lownr-fr}}$. Let

$$(Z,W,\phi,g)\in\mathbf{Corr}^{\text{lownr-fr}}(X,Y/V),(Z_0,\tilde{W}_0,\tilde{\phi}_0,\tilde{g}_0)\in\mathbf{Corr}^{\text{qfnr-fr}}(X_0,Y/V)$$

define an element in $\mathbf{Corr}^{\text{lownr-fr}}(X,Y/V)\times_{\mathbf{Corr}^{\text{lownr-fr}}(X_0,Y/V)}\mathbf{Corr}^{\text{qfnr-fr}}(X_0,Y/V)$. Since Y is smooth and consequently $Y\times\mathbb{A}^n$ is smooth, it follows by lemma 1 that there is an etale neighbourhood $U\to\mathbb{A}^n\times X$ of the closed subscheme $\tilde{W}_0\coprod_{W_0}W$, where $W_0=W\times_X X_0$, and a regular map $(\psi,r)\colon U\to\mathbb{A}^n\times Y$, such that

$$\psi\big|_{(U\times_X X_0)^{1\text{th}}_{\tilde{W}_0}}=\tilde{\phi}_0\big|_{(\mathbb{A}^n\times X_0)^{1\text{th}}_{\tilde{W}_0}},\quad \psi\big|_{U^{1\text{th}}_W}=\phi\big|_{(\mathbb{A}^n\times X)^{1\text{th}}_W},$$
$$r\big|_{\tilde{W}_0}=\tilde{g}_0\big|_{\tilde{W}_0},\qquad r\big|_W=g.$$

Using lemma 6 shrinking U we can assume that U is a complete intersection, i.e. that there is an immersion $U\to\mathbb{A}^m\times\mathbb{A}^n\times X$ and a set of regular functions $\phi_{n+1},\dots\phi_m\in\mathcal{O}(\mathbb{A}^m\times\mathbb{A}^n\times X)$ such that $U=Z(\phi_{n+1}\dots\phi_m)$, and moreover the restriction of ϕ_{n+i} on $(\mathbb{A}^m\times\mathbb{A}^n\times X_0)^{1\text{th}}_{\tilde{W}_0}\cup(\mathbb{A}^m\times\mathbb{A}^n\times X)^{1\text{th}}_Z$ is equal to the i-th coordinate function on $\mathbb{A}^m$.

Define $\tilde{W}=Z(\psi)\subset U\subset\mathbb{A}^{m+n}\times X$. Choose a functions $\tilde{\phi}_1,\dots\tilde{\phi}_n\in\mathcal{O}((\mathbb{A}^{m+n}\times X)^{1\text{th}}_{\tilde{W}})$ such that the restriction of $\tilde{\phi}_i$ on $(\mathbb{A}^{m+n}\times X_0)^{1\text{th}}_{\tilde{W}_0}\cup(\mathbb{A}^{m+n}\times X)^{1\text{th}}_Z$ is equal to the inverse image of ψ along the composition

$$(\mathbb{A}^{m+n}\times X_0)^{1\text{th}}_{\tilde{W}_0}\cup(\mathbb{A}^{m+n}\times X)^{1\text{th}}_Z\xrightarrow{pr}(\mathbb{A}^n\times X_0)^{1\text{th}}_{\tilde{W}_0}\cup(\mathbb{A}^n\times X)^{1\text{th}}_Z\to U$$

where pr is induced by the projection $\mathbb{A}^{m+n}\to\mathbb{A}^n$ and the second morphism is the canonical immersion.

Now define the required correspondence as

$$(\tilde{W},Z,\tilde{\phi}_1,\dots\tilde{\phi}_n,\phi_{n+1},\dots\phi_m,r\big|_{\tilde{W}})\in\mathbf{Corr}^{\text{qfnr-fr}}(X,Y/V).$$

□

Lemma 5. *Let S be a scheme and $f\colon F\to G$ be a morphism on $\mathbf{Pre}(\mathrm{Sm}_S)$. Assume that f satisfy the lifting property with respect to the closed immersions on affines. Then f in $\mathbb{A}^1$-equivalence in $\mathbf{Pre}(\mathrm{AffSm}_S)$, and f is a motivic equivalence in $\mathbf{Pre}(\mathrm{Sm}_S)$.*

Proof. See the arguments of [5]. □

Lemma 6. *Let X be an affine scheme, $Z\subset X$ be a closed subscheme, and $U\to X$ be an etale neighbourhood of Z. Then there is*

(1d) an open subscheme U', $U'\supset Z$,
(2d) a closed immersion $i\colon U'\to\mathbb{A}^m\times X$,
(3d) a set of regular functions $\phi_1,\dots\phi_m\in\mathcal{O}(\mathbb{A}^m\times X)$

such that

(1s) $i\colon U'\simeq Z(\phi_1,\dots\phi_m)$,
(2s) the immersion $X^{1\text{th}}_Z\to(\mathbb{A}^m\times X)$ induced by i and the isomorphism $U^{1\text{th}}_Z\simeq X^{1\text{th}}_Z$ is equal to the composition of the maps $X^{1\text{th}}_Z\to X\simeq 0\times X\to\mathbb{A}^m\times X$.

(3s) $\phi_i\big|_{(\mathbb{A}^m\times X)^{\mathrm{1th}}_Z} = t_i$*, where* t_i *denotes the* i*-th coordinate function on* $\mathbb{A}^m$*.*

Proof. Since the canonical map $U\to X$ is quasi-finite and X is affine, V is quasi-affine. So shrinking U we can make it be affine, and get the immersion $i'\colon U\to\mathbb{A}^{m-1}\times X$ such that the canonical map $U\to X$ is equal to the one induced by the projection $\mathbb{A}^{m-1}\times X\to X$. Then i' is given by the set of regular functions $(i'_1,\dots \imath'_{m-1})$ on U.

Choose any lift $r_j\in\mathcal{O}(\mathbb{A}^{m-1}\times X)$ of the functions $i'^*(t_j)$ where t_j denote the coordinate functions $j=1,\dots m-1$. Changing i'_j to $i'_j - i'^*(r_j)$ we get points **(1d), (2d), (1s), (2s)** with $m-1$ instead of m.

Let $e_1,\dots e_{m-1}, d\in\mathcal{O}(\mathbb{A}^{m-1}\times X)$ be functions such that

$$\begin{array}{ll} Z(e_1,\dots e_{m-1})\supset V, & Z(q)\supset Z(e_1,\dots e_{m-1})-V,\\ e_i\big|_{(\mathbb{A}^{m-1}\times X)^{\mathrm{1th}}_{(0\times Z)}} = t_i, & d\big|_{(\mathbb{A}^{m-1}\times X)^{\mathrm{1th}}_{(0\times Z)}} = 1. \end{array}$$

Now consider the immersion $\mathbb{A}^{m-1}\times X\to\mathbb{A}^m\times X$. Define $\phi_1,\dots\phi_{m-1}$ as the inverse image of $e_1,\dots e_{m-1}$ along the projection $\mathbb{A}^m\times X\to\mathbb{A}^{m-1}\times X$. Define $\phi_m = t_m d - 1\in\mathcal{O}(\mathbb{A}^m\times X)$, where $d\in\mathcal{O}(\mathbb{A}^m\times X)$ denote the inverse image of d along the mentioned above projection. □

Proposition 6. *Let* $Y\in\mathrm{AffSm}$*,* $V\subset Y$ *is open, then the morphism* $Y^{\text{tg-fr}}/V^{\text{tg-fr}}\to\langle Y/V\rangle^{\text{tg-fr}}$ *induces an equivalence in* $\mathbf{Pre}^{\mathrm{fr}}_{\mathrm{nis}}(S)$*.*

Proof. The claim follows by proposition 8 and lemma 7. □

Proposition 7. *The morphism* $Y^{\text{nis-fr}}/V^{\text{nis-fr}}\to\langle Y/V\rangle^{\text{nis-fr}}$ *induces an equivalence in* $\mathbf{Pre}_{\mathrm{nis},\mathbb{A}^1}(S,Z)$*.*

Proof. Proposition **??** implies that the morphism $Y^{\text{tg-fr}}/V^{\text{tg-fr}}\to\langle Y/V\rangle^{\text{tg-fr}}$ induces an equivalence in $\mathbf{Pre}_{\mathrm{nis},\mathbb{A}^1}(S,Z)$. Then the claim follows by the propositions 4, 5. □

Proposition 8. *There is an equivalence* $\mathbf{Pre}^{\text{tg-fr}}_{\Sigma}\simeq\mathbf{Pre}^{\text{CMon,tg-fr}}_{\Sigma}$ *given by the functor and the forgetful one.*

Lemma 7. *The morphism* $Y^{\text{tg-fr}}/V^{\text{tg-fr}}\to\langle Y/V\rangle^{\text{tg-fr}}$ *induces the equivalence* $v^*(Y^{\text{tg-fr}})/v^*(V^{\text{tg-fr}})\simeq v^*(\langle Y/V\rangle^{\text{tg-fr}})$ *in* $\mathbf{Pre}^{\text{tg-fr}}_{\mathrm{CMon,nis}}$*.*

Proof. Let $U=X_x$ be a local henselian scheme for $X\in\mathrm{Sm}_S$. Then the first statement of lemma 8 gives us the equivalence $v^*(Y^{\text{tg-fr}})(U)\simeq v^*(V^tgfr)(U)\oplus v^*(\langle Y/V\rangle^{\text{tg-fr}})(U)$, and the second one implies that the equivalence is the inverse morphism to the morphism $v^*(V^{\text{tg-fr}})(U)\oplus v^*(\langle Y/V\rangle^{\text{tg-fr}})(U)\to v^*(Y^{\text{tg-fr}})(U)$ induced by j and i. □

Lemma 8. *For any* $Y\in\mathrm{Sm}_S$*, open* $V\subset Y$ *and a local henselian* $U=X^h_x$*,* $X\in\mathrm{Sm}_S$*,* $x\in X$*, the sequence* $v^*(Y^{\text{tg-fr}})(U)\to v^*(V^{\text{tg-fr}})(U)\to v^*(\langle Y/V\rangle)^{\text{tg-fr}}(U)$ *splits into the equivalence*

$$p\colon v^*(Y^{\text{tg-fr}})(U)\simeq v^*(V^{\text{tg-fr}})(U)\oplus v^*(\langle Y/V\rangle)^{\text{tg-fr}}(U). \tag{7}$$

Proof. Define the subpresheaf $\langle Y/V\rangle^{\mathrm{essfintg}}\text{-fr}\subset\langle Y/V\rangle^{\text{tg-fr}}$ spanned by the correspondences (W,f,ϕ,g) such that f is finite and for each disjoint component $W'\subset W$, $W'\times_Y(Y\setminus V)\neq\emptyset$. Then we have the morphisms

$$\begin{array}{lllllll} j^{Y\setminus V} & : & \langle Y/V\rangle^{\mathrm{essfintg}}\text{-fr} & \to & Y^{\text{tg-fr}} & : & (W,f,\phi,g) \mapsto (W,f,\phi,g)\\ e & : & \langle Y/V\rangle^{\mathrm{essfintg}}\text{-fr} & \to & \langle Y/V\rangle^{\text{tg-fr}} & : & (W,f,\phi,g) \mapsto (W,f,\phi,g) \end{array}$$

(Step 1) Define the morphisms

$$(-)^V\colon Y^{\text{tg-fr}}\to V^tgfr, (-)^{Y\setminus V}\colon Y^{\text{tg-fr}}\to\langle Y/V\rangle^{\mathrm{essfintg}}\text{-fr}$$

as follows. For any quasi-finite syntomic span(W,f,g) by lemma 9 we have

$$W = W^V\amalg W^{Y\setminus V}, \mathbf{K}(W)\simeq\mathbf{K}(W^V)\times\mathbf{K}(W^{Y\setminus V}), \tag{8}$$

where the equivalence is provided by the morphisms which we denote as $(-)\big|_{W^V}\colon K(W)\simeq \mathbf{K}(W^V)$ and $(-)\big|_{W^{Y\setminus V}}\colon \mathbf{K}(W)\simeq \mathbf{K}(W^{Y\setminus V})$. Denote by $g'\colon W^V\to V$ the lift of $g\big|_{W^V}\colon W^V\to Y$. Then we have the morphisms

$$\begin{array}{llllll}(-)^V\colon Y^{\text{tg-fr}} & \to & V^{\text{tg-fr}} & : & (W,f,\phi,g) & \mapsto & (W^V, f\big|_{W^V}, \phi\big|_{W^V}, g') \\ (-)^{Y\setminus V} & \to & Y^{\text{tg-fr}}\to\langle Y/V\rangle^{\text{tg-fr}} & : & (W,f,\phi,g) & \mapsto & (W^{Y\setminus V}, f\big|_{W^{Y\setminus V}}, \phi\big|_{W^{Y\setminus V}}, g\big|_{W^{Y\setminus V}})\end{array}$$

We claim that the morphisms

(9)
$$\begin{array}{lllllll} q\colon Y^{\text{tg-fr}}(U) & \to & V^{\text{tg-fr}}(U)\oplus\langle Y/V\rangle^{\text{essfintg}}\text{-fr} & : & c & \mapsto & (c^V, c^{Y\setminus V}) \\ q^{-1}\colon V^{\text{tg-fr}}(U)\oplus\langle Y/V\rangle^{\text{essfintg}}\text{-fr}(U) & \to & Y^{\text{tg-fr}}(U) & : & (c_1,c_2) & \mapsto & c_1+c_2\end{array}$$

induces the pair of the inverse equivalences $v^*(q)$ and $v^*(q^{-1})$. The claim follows because of the equalities

$$q\circ i\simeq id_{\langle Y/V\rangle^{\text{essfintg}}\text{-fr}(U)}, q\circ j'\simeq id_{V^{\text{tg-fr}}},\ \text{where } j'=v^*(j)(U),$$

provided immediately by the definitions, and the equivalence

$$id_{v^*(Y^{\text{tg-fr}})(U)}\simeq (-)^V+(-)^{Y\setminus V}\colon c\simeq j(c^V)+i(c^{Y\setminus V}) \tag{10}$$

provided by (8).

Let us explain equivalence (10) in detail. Consider firstly the presheaf $v^*(\langle Y/V\rangle^{\text{syncor}})$. Since Y^{syncor} is the presheaves of the classical pointed sets, $v^*(Y^{\text{syncor}})$ is the presheaf of classical commutative monoids. So (8) implies the classical equality

$$id_{v^*(Y^{\text{syncor}})(U)}=(-)^V+(-)^{Y\setminus V} \tag{11}$$

. Then because of the equivalence $K(W)\simeq K(W^V)\times K(W^{Y\setminus V})$ equality (11) lifts to equivalence (10).

(Step 2)

Similarly to the morphism $(-)^{Y\setminus V}$ (at the beginning of the Step 1) define the morphism

$$(-)^{\text{ess}}\colon\langle Y/V\rangle^{\text{tg-fr}}\to\langle Y/V\rangle^{\text{essfintg}}\text{-fr}\colon (W,f,\phi,g)\mapsto (W^{Y\setminus V}, f\big|_{W^{Y\setminus V}}, \phi\big|_{W^{Y\setminus V}}, g\big|_{W^{Y\setminus V}})$$

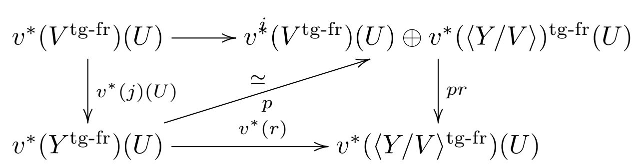
Then arguing like as in the Step 1 we get that $v^*((-)^{\text{ess}})$ and $v^*(e)$ is the pair of the inverse equivalences.

(Step 3)

By Step 1 and 3 we have the equivalences

$$\begin{array}{lllllll} v^*(q) & : & v^*(V^{\text{tg-fr}})(U) & \simeq & v^*(V^{\text{tg-fr}})(U)\oplus v^*(\langle Y/V\rangle^{\text{essfintg}}\text{-fr})(U) & : & v^*(q^{-1}), \\ v^*(j^{Y\setminus V}) & : & v^*(\langle Y/V\rangle^{\text{essfintg}}\text{-fr})(U) & \simeq & v^*(\langle Y/V\rangle^{\text{tg-fr}})(U) & : & v^*((-)^{\text{ess}})\end{array}$$

Note that by the construction we have the equivalence $(-)^{Y\setminus V}\simeq (r(-))^{\text{ess}}$ Hence we get the commutative diagram of morphisms of commutative monoids



where $p=(\mathrm{id}_{v^*(V^{\text{tg-fr}})(U)}\oplus v^*(e)\circ v^*(q)$ $j\colon V^{\text{tg-fr}}\to Y^{\text{tg-fr}}$ and $r\colon Y^{\text{tg-fr}}\to\langle Y/V\rangle^{\text{tg-fr}}$, and i and pr are the canonical morphisms. □

Lemma 9. *Let U be a local henselian scheme, Y be a scheme and $V\subset Y$ be an open subscheme. Then for any diagram $U\xleftarrow{f}W\xrightarrow{g}Y$ with f quasi-finite there is a splitting*

$$W=W^V\amalg W^{Y\setminus V},$$

where $W^V, W^{Y\setminus V} \subset W$ *open-closed subschemes such that*

$$W^V \times_Y V = W \times_Y V, W^{Y\setminus V} \times_Y Y \setminus V = W \times_Y^{Y\setminus V}.$$

Proof. By Zariski's main theorem f passes throw the composition $W \hookrightarrow \overline{W} \xrightarrow{\overline{f}} U$, with the first morphism being an open immersion and $\overline{f}$ being finite. Then since U is local henselian $\overline{W} = \coprod_i \overline{W}_i$ with $\overline{W}_i$ being local henselian. Set

$$\overline{W}^V = \coprod_{i, \overline{W}_i \times Y V \neq \emptyset} \overline{W}_i, \overline{W}^V = \coprod_{i, \overline{W}_i \times Y V = \emptyset} \overline{W}_i$$

Finally, set $W^V = \overline{W}^V \times_{\overline{W}} W$, $W^{Y\setminus V} = \overline{W}^{Y\setminus V} \times_{\overline{W}} W$. □

Corollary 5. *Finite normally framed correspondences are motivically equivalent to the quasi-finite ones. Finite tangentially framed correspondences are equivalent to the quasi-finite ones.*

8. Appendix 2: The stable motivic homotopy category with GW coefficients

Lemma 10. *For any scheme S there is an equivalence* $\mathbf{SH}(S, \mathrm{GW}_{\mathrm{char}}) \simeq \mathbf{SH}(S, \mathbb{Z}_{\mathrm{char}})$*, where* $\mathbb{Z}_{\mathrm{char}}(U) = \mathbb{Z}[p^{-1} \colon p = \mathrm{char}\mathcal{O}(x), x \in U]$ $\mathrm{GW}_{\mathrm{char}}(U) = \mathrm{GW}(U)[p_\epsilon^{-1} \colon p = \mathrm{char}\mathcal{O}(x), x \in U]$.

Proof. We start with the equivalence $\mathbf{SH}(S, \mathrm{GW}) \simeq \mathbf{SH}(S, \mathbb{Z})$ given by the definition of the left side. Then we have the equivalence $\mathbf{SH}(S, \mathrm{GW}'_{\mathrm{char}}) \simeq \mathbf{SH}(S, \mathbb{Z}_{\mathrm{char}})$, where $\mathrm{GW}'_{\mathrm{char}}(U) = \mathrm{GW}(U)[p^{-1} \colon p = \mathrm{char}\mathcal{O}(x), x \in U]$. Now to prove the claim it is enough to show that $\mathrm{GW}_{\mathrm{char}} \simeq \mathrm{GW}'_{\mathrm{char}}$. The cliam follows by two isomorphisms

$$\mathrm{GW}_{\mathrm{char}}/2_\epsilon \simeq \mathrm{GW}'_{\mathrm{char}}/2_\epsilon, \mathrm{GW}_{\mathrm{char}}[2_\epsilon^{-1}] \simeq \mathrm{GW}'_{\mathrm{char}}[2_\epsilon^{-1}]$$

where the fist one is because of the equality $p = p_\epsilon \in \mathrm{GW}_{\mathrm{char}}/2_\epsilon$ and the second one is because of $p \cdot 2_\epsilon = p_\epsilon 2_\epsilon \in \mathrm{GW}_{\mathrm{char}}$. □

9. Appendix 3: Σ-localisation

The subcategory of radditive (simplicial) presheaves $\mathbf{Pre}_\Sigma$ in the category of (simplicial) presheaves $\mathbf{Pre}$ is not the subcategory of sheaves with respect to a Grothendieck topology, but the behaver is similar. In the appendix we discuss how to fit it in to the formalism of the Grothendieck topological localisation functors.

To do this we can forget the symmetric monoidal structure on the category Sch given by direct colimits and attach coproducts back in a free way. So we consider an initial object in the category of adjunctions $\coprod \colon \mathrm{Sch}^\Sigma \to \mathrm{Sch} \colon u$ with Sch^Σ being a symmetirc monoidal category, and $\coprod$ being a functor of symmetric monoidal categories, and u being a functor (not preserving the symmetric monoidal structure) Define the topology Σ on Sm^Σ generated by the coverings $\mathcal{X} \to \coprod(\mathcal{X})$ and closed under the operations of the universal symmetrc monoidal stracture on Sm^Σ. Then the adjunction $\mathbf{Pre}(\mathrm{Sch}) \leftrightarrows \mathbf{Pre}_\Sigma(\mathrm{Sch})$ is equivalent to the admissible localisation $\mathbf{Pre}_\Sigma(\mathrm{Sch}^\Sigma) \leftrightarrows \mathbf{Sh}_\Sigma(\mathrm{Sch}^\Sigma)$, where $\mathbf{Pre}_\Sigma(\mathrm{Sch}^\Sigma)$ is denotes the subcategory in the category of functors $\mathrm{Sch}^\Sigma \to \mathrm{SSet}$ radditive with respect to the universal Σ-monoidal structure and $\mathbf{Sh}_\Sigma(\mathrm{Sch})$ is the category of sheaves with respect to the Σ-topology on Sch^Σ.

Let us do this with the elementary terms:

Definition 7. Let Sch be a classical category of schemes over a pairs S, $Z \subset S$. Denote by Sch^Σ the category of families of schemes, defined as follows. Sch^Σ is a classical category with objects being the pairs (A, X), $A \in \mathrm{Set}$ is a classical set, and $X \colon A \to \mathrm{Sch} \colon \alpha \mapsto X_\alpha$ be a map form the set A to the category Sch. The morphisms in Sch^Σ is a pair $(s, f) \colon (A, X) \to (B, Y)$ with $s \colon A \to B$ and f is a family of morphisms of schemes $f_{\alpha,\beta} \colon X_\alpha \to Y_\beta$ for each pair $\alpha \in A$, $\beta \in B$, $s(\alpha) = \beta$.

For a scheme $X \in \mathrm{Sch}$ denote by $(1, X) \in \mathrm{Sch}^\Sigma$ the object given by the set with one element and a scheme X.

Then we have the adjunction $\coprod \mathrm{Sch}^\Sigma \to \mathrm{Sch}\colon u$ with $\coprod\colon \mathrm{Sch}^\Sigma \to \mathrm{Sch}\colon (A, X) \mapsto \coprod_{\alpha\in A} X_\alpha$, $u\colon \mathrm{Sch} \to \mathrm{Sch}^\Sigma\colon X \mapsto (1, X)$.

Definition 8. Denote the Σ-topology on Sch^Σ as the topology with the coverings $(s, f)\colon (A, X) \to (B, Y)$ being the morphisms such that $\forall\beta \in B$, $Y_\beta \simeq \coprod_{\alpha\in s^{-1}(\beta)} X_\alpha$.

Denote by $\mathbf{Pre}_\Sigma(\mathrm{Sch}^\Sigma_S)$ the subcategory in the category of functors $Sch^\Sigma_S \to \mathrm{SSet}_\bullet$ spanned by the functors F such that $F(A, X) \simeq \prod_{\alpha\in A} F(1, X_\alpha)$. Denote by $\mathbf{Sh}_\Sigma(\mathrm{Sch}^\Sigma_S)$ the subcategory of $\mathbf{Pre}_\Sigma(\mathrm{Sch}^\Sigma_S)$ spanned by the sheaves in Σ-topology.

We call by the Σ-equivalences the local equivalences with respect to the Σ-topology on Sch^Σ.

Proposition 9. *The functor $\mathbf{Pre}_\Sigma(\mathrm{Sm}_S) \to \mathbf{Pre}(\mathrm{Sm}_S)$ is equivalent to the functor $\mathbf{Sh}_\Sigma(\mathrm{Sm}^\Sigma_S) \to \mathbf{Pre}_\Sigma(\mathrm{Sm}^\Sigma_S)$, and the last functor fits into the adjunction $L\colon \mathbf{Pre}_\Sigma(\mathrm{Sm}^\Sigma_S) \leftrightarrows \mathbf{Sh}_\Sigma(\mathrm{Sm}^\Sigma_S)\colon I$ with L being accessible localisation.*

Proof. Let us skip the first claim. The second claim is a particular case of a general localisation with respect the local equivalences for a Grothendieck topology. □

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