Higher limits. fr-codes

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Aim

- I want to present a connection

$$\text{(homological algebra)} \longleftrightarrow \text{(combinatorial group theory)}$$

- Groups here can be replaced by any algebraic object. Historically associative algebras were the first.
- Hopf's formula: if $G = F/R$, where F is a free group,

$$H_2(G) = \frac{R \cap [F,F]}{[R,F]} = \mathsf{lim}\ \frac{R}{[R,F]}.$$

- $\frac{R\cap[F,F]}{[R,F]}$ is the largest part of $\frac{R}{[R,F]}$ which is independent of the choice of F and R.
- The aim is to explain the formula with lim and show how to generalise it.

Limits.

- Limit $\lim \mathcal{F}$ of a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is an object of $\mathcal{D}$ together with a universal collection of morphisms $\{\varphi_c : \lim \mathcal{F} \to \mathcal{F}(c)\}_{c\in\mathcal{C}}$ such that $\mathcal{F}(f)\varphi_c = \varphi_{c'}$ for every morphism $f : c \to c'$.
- Universality means that for every object $d \in \mathcal{D}$ and every collection of morphisms $\{\psi_c : d \to \mathcal{F}(c)\}_{c\in\mathcal{C}}$ such that $\mathcal{F}(f)\psi_c = \psi_{c'}$ for every morphism $f : c \to c'$ there exists a unique morphism $\alpha : d \to \lim \mathcal{F}$ such that $\psi_c = \varphi_c\alpha$.
- If a limit exists, it is unique up to a unique isomorphism.

Limits over strongly connected categories

- Let k be a commutative ring and $\mathcal{C}$ be a category.
- $\mathcal{C}$ is **strongly connected** if $\mathcal{C}(c,c') \neq \varnothing$ for any $c,c' \in \mathcal{C}$.

Proposition

Let $\mathcal{C}$ be a strongly connected category and $\mathcal{F} : \mathcal{C} \to \mathsf{Mod}(k)$ be a functor. Then $\lim \mathcal{F}$ exists, for any $c \in \mathcal{C}$ the morphism

$$\varphi_c : \lim \mathcal{F} \rightarrowtail \mathcal{F}(c)$$

is a monomorphism and $\lim \mathcal{F}$ is the largest constant subfunctor of $\mathcal{F}$.

- Roughly speaking, in this case $\lim \mathcal{F}$ consists of elements of $\mathcal{F}(c)$ that are independent of c.

Higher limits

- Let k be a commutative ring and $\mathcal{C}$ be a category.
- All limits of all functors $\mathcal{C} \to \mathsf{Mod}(k)$ exist.
  (If we consider big enough universe)
- We get the functor

$$\mathsf{lim} : \mathsf{Funct}(\mathcal{C}, \mathsf{Mod}(k)) \to \mathsf{Mod}(k),$$

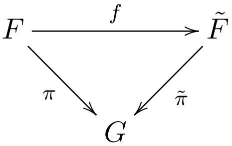
$$\mathcal{F} \mapsto \mathsf{lim}\ \mathcal{F}.$$

- $\mathsf{lim} : \mathsf{Funct}(\mathcal{C}, \mathsf{Mod}(k)) \to \mathsf{Mod}(k)$ is a left exact functor between abelian categories.
- **Higher limits** of $\mathcal{F} : \mathcal{C} \to \mathsf{Mod}(k)$ are defined as follows:

$$\mathsf{lim}^i\, \mathcal{F} := \mathbf{R}^i \,\mathsf{lim}\, \mathcal{F}.$$

The category of presentations of a group

- Let G be a group.
- A **presentation** of G is an epimorphism from a free group $\pi : F \twoheadrightarrow G$.
- If $R = \mathrm{Ker}(\pi)$, then $G \cong F/R$.
- A morphism of presentations $f : (\pi : F \twoheadrightarrow G) \to (\tilde{\pi} : \tilde{F} \twoheadrightarrow G)$ is a homomorphism $f : F \to \tilde{F}$ such that $\tilde{\pi} f = \pi$.



- $\mathsf{Pres}(G)$ is the **category of presentations** of G.
- $\mathsf{Pres}(G)$ is strongly connected.
- If A is an **associative algebra** over a field k, the category $\mathsf{Pres}(A)$ is defined similarly.
- Further all limits are taken over the category of presentations.

The origin of the approach: Quillen's theorem about cyclic homology

- Let A be an algebra over a field k. If $F \twoheadrightarrow A$ is a presentation of A, we set $\mathbf{r} := \mathrm{Ker}(F \twoheadrightarrow A)$.
- For an F-bimodule M we set

$$M_\natural = \frac{M}{[M,F]} = HH_0(F,M),$$

where $[M,F]$ is the vector space generated by elements $mf - fm$.

Theorem (Quillen (1989))

Let A be an algebra over a field k of characteristic 0. Then even cyclic homology are isomorphic to the limits

$$HC_{2n}(A) \cong \lim\ (F/\mathbf{r}^{n+1})_\natural.$$

- How to present odd cyclic homology on this language?
- Our answer: use higher limits.

Odd cyclic homology as $\lim^1$

Theorem (Quillen (1989))

Let A be an algebra over a field k of characteristic 0. Then there are isomorphisms

$$HC_{2n}(A) \cong \lim^0 \ (F/\mathbf{r}^{n+1})_\natural.$$

Theorem (R. Mikhailov, – (2013))

Let A be an **augmented** *algebra over a field k of characteristic 0. Then there are isomorphisms*

$$HC_{2n-1}(A) \cong \lim^1 \ (F/\mathbf{r}^{n+1})_\natural.$$

- $\lim^1$ allows to present odd cyclic homology but only for augmented algebras.

Hochschild homology

Theorem (Quillen (1989))

Let A be an algebra over a field k of characteristic 0. Then there are isomorphisms

$$HC_{2n}(A) \cong \mathsf{lim}^0 \ (F/\mathbf{r}^{n+1})_\natural.$$

Theorem (R. Mikhailov, – (2013))

Let A be an algebra over a field k (of any characteristic), M be a A-bimodule, $n \geq 1$, $0 \leq i \leq n-1$. Then there are natural isomorphisms

$$HH_{2n-i}(A) \simeq \mathsf{lim}^i(\mathbf{r}^n/\mathbf{r}^{n+1})_\natural$$

$$HH_{2n-i}(A, M) \simeq \mathsf{lim}^i \ (\mathbf{r}^n/\mathbf{r}^{n+1}) \otimes_{A^e} M$$

- Higher limits allow to present odd Hochschild homology.

Connes-Tzygan exact sequence

Consider the short exact sequence

$$0 \longrightarrow \mathbf{r}^n/\mathbf{r}^{n+1} \longrightarrow F/\mathbf{r}^{n+1} \longrightarrow F/\mathbf{r}^n \longrightarrow 0.$$

Conjecture: This short exact sequence after applying $\mathsf{lim}^* \, (-)_\natural$ induces the Connes-Tzygan exact sequence:

$$\ldots \longrightarrow HH_{2n}(A) \longrightarrow HC_{2n}(A) \longrightarrow HC_{2n-2}(A) \longrightarrow HH_{2n-1}(A) \longrightarrow \ldots$$

Group homology

- Let G be a group.
- If $F \twoheadrightarrow G$ is a presentation, we set $R := \mathrm{Ker}(F \twoheadrightarrow G)$.
- R_{ab} is so-called relation module over G (functorial by $\mathsf{Pres}(G)$).

Theorem (R. Mikhailov, – (2013))

For a group G and a G-module M there are isomorphisms:

$$H_{2n-i}(G) = \mathsf{lim}^i (R_{ab}^{\otimes n})_G,$$

$$H_{2n-i}(G, M) = \mathsf{lim}^i \, R_{ab}^{\otimes n} \otimes_{\mathbb{Z}G} M, \qquad 0 \leq i \leq n-1.$$

Hopf's formula

- $H_{2n-i}(G) = \mathsf{lim}^i (R_{ab}^{\otimes n})_G$
- $n := 1, i := 0.$
- $(R_{ab})_G = \frac{R}{[F,R]}$
- $H_2(G) = \mathsf{lim}\ \frac{R}{[F,R]}$
- $H_2(G) = \frac{R \cap [F,F]}{[F,R]}$ (Hopf's formula).
- $H_2(G) = \frac{R \cap [F,F]}{[F,R]}$ is the biggest subgroup of $\frac{R}{[F,R]}$ which is 'independent' under the choice of $F \twoheadrightarrow G$.
- All this theory can be considered as a generalisation of the Hopf's formula.

fr-codes of functors $\mathsf{Gr} \to \mathsf{Ab}$

- Pres is the category whose objects are presentations of a group $F \overset{\pi}{\twoheadrightarrow} G$, and whose morphisms are commutative squares
$$\begin{array}{ccc} F_1 & \longrightarrow & F_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ G_1 & \longrightarrow & G_2. \end{array}$$
- The fibre of the forgetful functor $\mathsf{Pres} \to \mathsf{Gr}$ over a group G is the category $\mathsf{Pres}(G)$.
- $\mathbb{Z}F$ can be considered as a functor $\mathbb{Z}F : \mathsf{Pres} \to \mathsf{Ab}$.
- A functorial ideal is a subfunctor $\mathbf{x} \triangleleft \mathbb{Z}F : \mathsf{Pres} \to \mathsf{Ab}$ consisting of ideals.
- **Example.** The augmentation ideal $\mathbf{f} \triangleleft \mathbb{Z}F$ is a functorial ideal.
- **Example.** The ideal $\mathbf{r} = \mathrm{Ker}(\mathbb{Z}F \twoheadrightarrow \mathbb{Z}G)$ is a functorial ideal.
- **Example.** All combinations like $\mathbf{f}^2\mathbf{r} + \mathbf{r}^2\mathbf{f} + (\mathbf{rfrf} \cap \mathbf{f}^6) + \mathbf{r}^{10}$ are functorial ideals (we can use $\cdot, +, \cap$).

fr-codes of functors $\mathsf{Gr} \to \mathsf{Ab}$

- For a functorial ideal $\mathbf{x}$ we can consider higher limits

$$^{i}[\mathbf{x}] = \lim_{\mathsf{Pres}(G)}{}^{i}\, \mathbf{x} : \mathsf{Gr} \to \mathsf{Ab}.$$

$$G \mapsto \lim_{\mathsf{Pres}(G)}{}^{i}\, \mathbf{x}.$$

- If $\mathbf{x} \subseteq \mathbf{f}$, then ${}^{0}[\mathbf{x}] = 0$. Hence, the first interesting case is ${}^{1}[\mathbf{x}]$.
- We set

$$[\mathbf{x}] := {}^{1}[\mathbf{x}].$$

fr-codes of functors $\mathsf{Gr} \to \mathsf{Ab}$

Examples:
Let $I : \mathsf{Gr} \to \mathsf{Ab}$ be the functor that sends G to the augmentation ideal. Then

$$I(G) = [\mathbf{r}] = \lim{}^1 \mathbf{r}.$$

More examples:

$$G_{ab} = [\mathbf{r} + \mathbf{f}^2],$$
$$I \otimes_{\mathbb{Z}G} I = [\mathbf{fr} + \mathbf{rf}],$$
$$I^{\otimes_{\mathbb{Z}G} 3} = [\mathbf{f}^2\mathbf{r} + \mathbf{frf} + \mathbf{rf}^2],$$
$$I^2 \otimes_{\mathbb{Z}G} I = [\mathbf{f}^2\mathbf{r} + \mathbf{rf}],$$
$$H_4(G) = [\mathbf{fr}^2 + \mathbf{r}^2\mathbf{f}],$$
$$H_6(G) = [\mathbf{fr}^3 + \mathbf{r}^3\mathbf{f}],$$
$$\mathsf{Tor}(G_{ab}, G_{ab}) = [\mathbf{r}^2 + \mathbf{f}^3],$$

$$I/I^3 = [\mathbf{r} + \mathbf{f}^3],$$
$$G_{ab} \otimes G_{ab} = [\mathbf{fr} + \mathbf{rf} + \mathbf{f}^3],$$
$$(I/I^3)^{\otimes_{\mathbb{Z}G} 2} = [\mathbf{fr} + \mathbf{rf} + \mathbf{f}^4],$$
$$(I^2/I^4) \otimes_{\mathbb{Z}G} I = [\mathbf{f}^2\mathbf{r} + \mathbf{rf} + \mathbf{f}^5],$$
$$H_3(G) = [\mathbf{r}^2 + \mathbf{frf}]$$
$$H_5(G) = [\mathbf{r}^3 + \mathbf{fr}^2\mathbf{f}]$$
$$L_2 \bigotimes{}^3 G_{ab} = [\mathbf{r}^3 + \mathbf{f}^4],$$

fr-codes of functors Gr → Ab

- The class of functors that can be obtained as $[\mathbf{x}]$, where $\mathbf{x}$ is a 'polynomial' of $\mathbf{f}, \mathbf{r}$, is called **fr-universe**.
- $\mathbf{x}$ is called an **fr-code** of the functor.
- The class of functors that can be obtained as ${}^i[\mathbf{x}]$, where $\mathbf{x}$ is a 'polynomial' of $\mathbf{f}, \mathbf{r}$, is called **higher fr-universe**.
- There are a lot of functors in the **fr**-universe:

$$H_{2n+2}(G) = [\mathbf{f}\mathbf{r}^{n+1} + \mathbf{r}^{n+1}\mathbf{f}], \qquad H_{2n-1}(G) = [\mathbf{r}^n + \mathbf{f}\mathbf{r}^{n-1}\mathbf{f}] \quad n \geq 1.$$

$$I^l \otimes_{\mathbb{Z}G} I^{\otimes_{\mathbb{Z}G} n} = [\mathbf{r}\mathbf{f}^{n-1} + \sum_{i=1}^{n-1} \mathbf{f}^{l+i}\mathbf{r}\mathbf{f}^{n-i-1}] \quad \text{for } n, l \geq 1.$$

$$(I^l/I^k) \otimes_{\mathbb{Z}G} I^{\otimes_{\mathbb{Z}G} n} = [\mathbf{r}\mathbf{f}^{n-1} + \sum_{i=1}^{n-1} \mathbf{f}^{l+i}\mathbf{r}\mathbf{f}^{n-i-1} + \mathbf{f}^{k+1}] \quad \text{for } n, l \geq 1, k > l.$$

$$G_{ab}^{\otimes n} = [\sum_{i=1}^{n} \mathbf{f}^{i-1}\mathbf{r}\mathbf{f}^{n-i} + \mathbf{f}^{n+1}] \quad \text{for } n \geq 1.$$

- **Question**: is there an **fr**-code for $H_2(G)$?
- **Question**: is there an **fr**-code for $L_i\otimes^n G_{ab}$ for $1 \le i \le n-2$?
- There is a higher **fr**-code:

$$L_{n-i}\bigotimes^n G_{ab} \cong {}^i[\mathbf{r}^n + \mathbf{f}^{n+1}].$$

fr-code	$\lim^1$	$\lim^2$	$\lim^3$	$\lim^4$
f	0	0	0	0
r	I	0	0	0
rr	0	$I \otimes I$	0	0
rrr	0	0	$I \otimes I \otimes I$	0
rrrr	0	0	0	$I \otimes I \otimes I \otimes I$
fr+rf	$I \otimes_{\mathbb{Z}G} I$	0	0	0
ffr+frf+rff	$I \otimes_{\mathbb{Z}G} I \otimes_{\mathbb{Z}G} I$	0	0	0
r+ff	G_{ab}	0	0	0
r+fff	I/I^3	0	0	0
rf+ffr	$I^2 \otimes_{\mathbb{Z}G} I$	0	0	0
rf+fffr	$I^3 \otimes_{\mathbb{Z}G} I$	0	0	0
rfr+frr+ffff	$\mathrm{Tor}(G_{ab} \otimes G_{ab}, G_{ab})$	0	0	0
fr+rf+fff	$G_{ab} \otimes G_{ab}$	0	0	0
rff+frf+rff+ffff	$G_{ab} \otimes G_{ab} \otimes G_{ab}$	0	0	0
rr+fff	$\mathrm{Tor}(G_{ab}, G_{ab})$	$G_{ab} \otimes G_{ab}$	0	0
rrr+ffff	$L_2 \otimes^3 (G_{ab})$	$L_1 \otimes^3 (G_{ab})$	$G_{ab} \otimes G_{ab} \otimes G_{ab}$	0
rrrr+fffff	$L_3 \otimes^4 (G_{ab})$	$L_2 \otimes^4 (G_{ab})$	$L_1 \otimes^4 (G_{ab})$	$G_{ab}^{\otimes 4}$
rr+frf	$H_3(G)$	$I \otimes_{\mathbb{Z}G} I$	0	0
rrf+frr	$H_4(G)$	$H_3(G)$	$I \otimes_{\mathbb{Z}G} I$	0
rrr+frrf	$H_5(G)$	$H_4(G)$	$H_3(G)$	$I \otimes_{\mathbb{Z}G} I$
rrrf+frrr	$H_6(G)$	$H_5(G)$	$H_4(G)$	$H_3(G)$
rf+ffr+ffff	$I^2/I^3 \otimes G_{ab}$	0	0	0
rfff+rfr+rrf	0	$I \otimes G_{ab} \otimes G_{ab}$	0	0
rrfff+rrfr+rrrf	0	0	$I \otimes I \otimes G_{ab} \otimes G_{ab}$	0

Technique. Monoadditive representations

- Let $\mathcal{C}, \mathcal{D}$ be categories with pairwise coproducts and $\mathcal{F} : \mathcal{C} \to \mathcal{D}$.
- The morphisms $c_1 \overset{i_1}{\to} c_1 \sqcup c_2 \overset{i_2}{\leftarrow} c_2$ induce the morphism
$$\mathcal{F}(c_1) \sqcup \mathcal{F}(c_2) \longrightarrow \mathcal{F}(c_1 \sqcup c_2).$$

Def. $\mathcal{F}$ is **additive** (resp. **monoadditive, split monoadditive**) if this morphism is an isomorphism (resp. monomorphism, split monomorphism in the category of bifunctors).

- The functors $\mathsf{sq} : \mathcal{C} \to \mathcal{C}$ and $\mathsf{sq} : \mathcal{D} \to \mathcal{D}$ given by $\mathsf{sq}(x) = x \sqcup x$.
- Then we have $\mathsf{T}_{\mathcal{F}} : \mathsf{sq} \circ \mathcal{F} \longrightarrow \mathcal{F} \circ \mathsf{sq}$.
- A representation of $\mathcal{C}$ is a functor $\mathcal{C} \to \mathsf{Mod}(k)$.
- Let $\mathcal{F} : \mathcal{C} \to \mathsf{Mod}(k)$ be a monoadditive representation. Set $\Sigma\mathcal{F} := \mathrm{coker}(\mathsf{T}_{\mathcal{F}})$.
$$0 \longrightarrow \mathcal{F} \oplus \mathcal{F} \overset{\mathsf{T}_{\mathcal{F}}}{\longrightarrow} \mathcal{F} \circ \mathsf{sq} \longrightarrow \Sigma\mathcal{F} \longrightarrow 0$$

Def. A monoadditive representation $\mathcal{F}$ is said to be $\boldsymbol{n}$**-monoadditive**, if $\Sigma\mathcal{F}$ is $(n-1)$-monoadditive.

Technique. Monoadditive representations

Proposition

Let $\mathcal{F}$ be a monoadditive representation of $\mathcal{C}$. Then for any $n \geq 0$ there is an isomorphism:

$$\mathsf{lim}^n \mathcal{F} \cong \mathsf{lim}^{n-1} \Sigma\mathcal{F}.$$

Corollary

If $\mathcal{F}$ *is an* n*-monoadditive representation, then* $\mathsf{lim}^i \mathcal{F} = 0$ *for* $0 \leq i < n$ *and* $\mathsf{lim}^i \mathcal{F} = \mathsf{lim}^{i-n}\Sigma^n\mathcal{F}$.

Corollary

If $\mathcal{F}$ *is an* ∞*-monoadditive representation, then* $\mathsf{lim}^i\mathcal{F} = 0$ *for any* $i \geq 0$.

Technique. Monoadditive representations

Proposition

If $\mathcal{F}$ is a split monoadditive representation, then $\Sigma\mathcal{F}$ is a split monoadditive representation.

Corollary

A split monoadditive representation is ∞-monoadditive.

Technique. Monoadditive representations

split monoadditive $\Rightarrow$ ∞-monoadditive $\Rightarrow$ monoadditive

$\mathsf{lim}^* = 0$	$\mathsf{lim}^* = 0$	$\mathsf{lim}^0 = 0$
		$\mathsf{lim}^i\, \mathcal{F} = \mathsf{lim}^{i-1}\, \Sigma\mathcal{F}$

Example of a proof

- We can prove that $\mathbf{f}$ is split monoadditive.
- Hence $\lim^i \mathbf{f} = 0$ for all i.
- Consider the short exact sequence

$$0 \longrightarrow \mathbf{r} \longrightarrow \mathbf{f} \longrightarrow I \longrightarrow 0$$

- Consider the corresponding long exact sequence of higher limits

$$0 \to \lim^0 \mathbf{r} \to \lim^0 \mathbf{f} \to \lim^0 I \to \lim^1 \mathbf{r} \to \lim^1 \mathbf{f} \to \lim^1 I \to \dots$$

- I is a constant functor. It follows that $\lim^i I = 0$ for $i > 0$ and $\lim^0 I = I$.
- Hence $\lim^1 \mathbf{r} = I$ and $\lim^i \mathbf{r} = 0$ for $i \neq 1$.

$$I = [r].$$