Homology of R-completions of a group and finite R-bad spaces

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Plan

- R-completion of a group ($R = \mathbb{Z}$, $\mathbb{Z}/p$, $\mathbb{Q}$).
- Main results:
$$H_2(\hat{F}_R, R) \neq 0$$
for $R = \mathbb{Z}/p$, $\mathbb{Q}$.
- Secondary Motivation: Comparison homomorphism
$$H_2^{\mathsf{disc}}(\mathcal{G}, \mathbb{Z}/p) \to H_2^{\mathsf{cont}}(\mathcal{G}, \mathbb{Z}/p)$$
for a pro-p-group $\mathcal{G}$.
- Primary Motivation: R-good and R-bad spaces in sense of Bousfield-Kan.
- Sketch of the proof for $R = \mathbb{Z}/p$.
- Sketch of the proof for $R = \mathbb{Q}$.
- HR-localisation of a group.
- Primary Motivation': The Bousfield problem.
- HR-length of a group.

R-completion of a group ($R = \mathbb{Z}$, $\mathbb{Z}/p$, $\mathbb{Q}$).

- If $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ and $c \in \mathcal{C}$, then $\mathcal{D}$-completion of c is the (inverse) limit over the category of all morphisms $c \to d$, where $d \in D$.
$$\hat{c}_{\mathcal{D}} = \lim(c \downarrow \mathcal{D} \to \mathcal{D})$$
- **Example:** If $\mathcal{C} = \mathbf{Gr}$ and $\mathcal{D} = \mathbf{FinGr}$, then $\hat{G}_{\mathbf{FinGr}}$ is the profinite completion.
- Let $R \in \{\mathbb{Z}, \mathbb{Z}/p, \mathbb{Q}\}$
- A group N is R-nilpotent if there exist a finite central series
$$N = H_1 \supseteq H_2 \supseteq \cdots \supseteq H_k = 1$$
such that H_n/H_{n+1} is an R-module.
- The R-completion $\hat{G}_R$ of G is the completion with respect to the subcategory of R-nilpotent groups.
- The R-completion $\hat{G}_R$ in the world of groups corresponds to the R-completion in sense of Bousfield-Kan $R_\infty X$ in the world spaces.

R-completion of a group ($R = \mathbb{Z}$, $\mathbb{Z}/p$, $\mathbb{Q}$).

- $\mathbb{Z}$-completion = pronilpotent completion. It can be described as the limit
$$\hat{G}_{\mathbb{Z}} = \varprojlim G/\gamma_n(G),$$
where $\gamma_n(G)$ is the lower central series.
- If G is finitely generated, then
$$\hat{G}_{\mathbb{Z}/p} \cong \hat{G}_{\textsf{pro-}p},$$
where $\hat{G}_{\textsf{pro-}p}$ is the limit of G/H over all normal subgroups H such that G/H is a finite p-group.
- If F is a finitely generated free group, $\hat{F}_{\mathbb{Z}/p}$ is the free pro-p group.
- $$\hat{G}_{\mathbb{Q}} = \varprojlim G/\gamma_n(G) \otimes \mathbb{Q},$$
where $- \otimes \mathbb{Q}$ is Mal'cev completion of a nilpotent group.

R-completion of a group ($R = \mathbb{Z}$, $\mathbb{Z}/p$, $\mathbb{Q}$).

- Why R-completion of groups is useful?
- $\mathbb{Z}/p$-completion is obviously important in the theory of profinite groups.
- $R_\infty X$ denotes the Bousfield Kan R-completion of X which is originally defined using the cosimplicial construction $\tilde{R}X$..
- $$\mathrm{sSet}_{\mathrm{red}} \underset{\bar{W}}{\overset{G}{\rightleftarrows}} \mathrm{sGr}$$
- $$\mathsf{Ho}(\mathrm{sSet}_{\mathrm{red}}) \simeq \mathsf{Ho}(\mathrm{sGr})$$
- There is a definition of $R_\infty X$ via R-completion of groups:
$$R_\infty X \simeq \bar{W}(\widehat{GX}_R).$$
- (Keune 1973) Let A be a ring and $F_\bullet \to \mathsf{GL}(A)$ be the free simplicial resolution. Then the algebraic K-theory of A can be obtained from the $\mathbb{Z}$-completion of $F_\bullet$:
$$K_{n+1}(A) = \pi_n(\widehat{(F_\bullet)}_{\mathbb{Z}}).$$

Main results: $H_2(\hat{F}_R, R) \neq 0$ for $R = \mathbb{Z}/p$, $\mathbb{Q}$.

- Here we consider the completion $\hat{G}_R$ as a discrete group.
- F is a free group of rank ≥ 2.
- **Theorem** (Bousfield, 1977): $H_2(\hat{F}_{\mathbb{Z}}, \mathbb{Z})$ is uncounable.
- In the same text Bousfield asked:
  $H_2(\hat{F}_R, R) \overset{?}{=} 0$ for the fields $R = \mathbb{Z}/p, \mathbb{Q}$.
- **Theorem** (Bousfield, 1992):
  either $H_2(\hat{F}_{\mathbb{Z}/p}, \mathbb{Z}/p)$ or $H_3(\hat{F}_{\mathbb{Z}/p}, \mathbb{Z}/p)$ is uncounable.
- <u>**The Main Theorem**</u> (Mikhailov and I, 2017):
  $H_2(\hat{F}_R, R)$ is uncounable for $R \in \{\mathbb{Z}/p, \mathbb{Q}\}$.
- **Corollary**: There exist uncountably many non-isomorphic central extensions of the free pro-p-group $\mathcal{F} = \hat{F}_{\text{pro-}p}$:

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow E \longrightarrow \mathcal{F} \longrightarrow 1.$$

  It is interesting because there is no such a profinite extension.

Main results: $H_2(\hat{F}_R, R) \neq 0$ for $R = \mathbb{Z}/p$, $\mathbb{Q}$.

Theorem (Bousfield'77 + Mikhailov and I'17)

If F is the free group of rank ≥ 2, then

$$H_2(\hat{F}_R, R) \neq 0$$

for any $R \in \{\mathbb{Z}, \mathbb{Z}/p, \mathbb{Q}\}$.

Sec. Motiv.: Comparison $H^2_{\mathsf{cont}}(\mathcal{G}, \mathbb{Z}/p) \to H^2_{\mathsf{disc}}(\mathcal{G}, \mathbb{Z}/p)$

- Let $\mathcal{G}$ be a pro-p-group.
- If we denote by $\mathcal{G}^{\mathsf{disc}}$ the same group with discrete topology, we get a homomorphism of topological groups

$$\mathcal{G}^{\mathsf{disc}} \longrightarrow \mathcal{G}.$$

- Then we obtain a comparison homomorphism

$$\varphi^2 : H^2_{\mathsf{cont}}(\mathcal{G}, \mathbb{Z}/p) \longrightarrow H^2_{\mathsf{disc}}(\mathcal{G}, \mathbb{Z}/p)$$

- **Question** (Fernandez-Alcober, Kazatchkov, Remeslennikov, Symonds, 2008):
  Does there exist a finitely presented pro-p group $\mathcal{G}$ for which φ^2 is not an isomorphism?
- **Answer** (Mikhailov and I, 2017): Yes. $\mathcal{G} = \mathcal{F} = \hat{F}_{\mathsf{pro}-p}$.

$$H^2_{\mathsf{cont}}(\mathcal{F}, \mathbb{Z}/p) = 0, \qquad H^2_{\mathsf{disc}}(\mathcal{F}, \mathbb{Z}/p) \neq 0.$$

Sec. Motiv.: Comparison $H^2_{\mathsf{cont}}(\mathcal{G},\mathbb{Z}/p) \to H^2_{\mathsf{disc}}(\mathcal{G},\mathbb{Z}/p)$

- (Non-)dually for a profinite group $\mathcal{G}$ one can define homological version of the homomorphism

  $$\varphi_2 : H_2^{\mathsf{disc}}(\mathcal{G},\mathbb{Z}/p) \longrightarrow H_2^{\mathsf{cont}}(\mathcal{G},\mathbb{Z}/p)$$

  and ask the same questions.
- **Warning:** discrete homology and cohomology are dual in the discrete sense; continuous homology and cohomology are dual in the continuous sense. So φ_2 is not dual to φ^2 in any sense.
- **Theorem**(R.Mikhailov and I, 2017). For a pro-p-presentation $\mathcal{G} = \mathcal{F}/\mathcal{R}$ of a finitely generated pro-p-group $\mathcal{G}$ there is an exact sequence

  $$H_2^{\mathsf{disc}}(\mathcal{F},\mathbb{Z}/p) \longrightarrow H_2^{\mathsf{disc}}(\mathcal{G},\mathbb{Z}/p) \xrightarrow{\varphi_2} H_2^{\mathsf{cont}}(\mathcal{G},\mathbb{Z}/p) \longrightarrow 0,$$

  where $\mathcal{F} = \hat{F}_{\mathbb{Z}/p}$.

Prim. Motiv.: R-good and R-bad spaces.

- This theorems were motivated by the Bousfield-Kan theory of R-completion $R_\infty X$ of arbitrary space X.

$$X \longrightarrow R_\infty X.$$

- $\mathbb{Z}/p$-completion coincides with the p-profinite completion of Sullivan for simply connected spaces with $H_*(X, \mathbb{Z}/p)$ of finite type.
- They were interested in $R_\infty X$ because $\pi_* R_\infty X$ served as the target of the unstable Adams spectral sequence.
- X is **R-good** if $X \to R_\infty X$ is an R-homology equivalence.
- A space X is **R-good** iff $R_\infty(R_\infty X) = R_\infty X$ and iff $X \to R_\infty X$ is the R-homological localisation of X.

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "R\text{-homology equivalence}"'] & R_\infty X \\ & Y \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

Prim. Motiv.: R-good and R-bad spaces

- Bousfield-Kan proved the following (1972):
  1. 1-connected spaces are R-good for any R.
  2. Nilpotent spaces are R-good for any R.
  3. The infinite wedge of circles $\bigvee_{i=1}^{\infty} S^1$ is R-bad for any R.
  4. $R_\infty K(F,1) = K(\hat{F}_R, 1)$.
- Then $\bigvee_{i=1}^{k} S^1$ is R-good iff $H_n(\hat{F}_R, R) = 0$ for $n \geq 2$.
- (Bousfield, 1977) $S^1 \vee S^1$ is $\mathbb{Z}$-bad,
  because $H_2(\hat{F}_{\mathbb{Z}}, \mathbb{Z}) \neq 0$.
- They had a conjecture that finite spaces are good over fields.
- (Bousfield, 1992) $S^1 \vee S^1$ is $\mathbb{Z}/p$-bad,
  because either $H_2(\hat{F}_{\mathbb{Z}/p}, \mathbb{Z}/p) \neq 0$ or $H_3(\hat{F}_{\mathbb{Z}/p}, \mathbb{Z}/p) \neq 0$.
- (Mikhailov and I, 2017) $S^1 \vee S^1$ is $\mathbb{Q}$-bad,
  because $H_2(\hat{F}_{\mathbb{Q}}, \mathbb{Q}) \neq 0$.

Prim. Motiv.: R-good and R-bad spaces

Theorem (Bousfield'77+Bousfield'92+ Mikhailov and I'17)

$S^1 \vee S^1$ is R-bad for any $R \in \{\mathbb{Z}, \mathbb{Z}/p, \mathbb{Q}\}$.

In all 3 cases it was the first known example of a finite R-bad space.

Sketch of the proof for $R = \mathbb{Z}/p$.

- Hopf like formulas:
- (Hopf's formula) If $H \lhd F$, then

$$H_2(F/H) = \frac{H \cap [F,F]}{[H,F]}$$

- If G is a group and $H \lhd G$, then

$$H_2(G) \longrightarrow H_2(G/H) \longrightarrow \frac{H \cap [G,G]}{[H,G]} \longrightarrow 0$$

$$H_2(G,\mathbb{Z}/p) \longrightarrow H_2(G/H,\mathbb{Z}/p) \longrightarrow \frac{H \cap [G,G]G^p}{[H,G]H^p} \longrightarrow 0$$

- If $\mathcal{G}$ is a profinite group and $\mathcal{H}$ is a normal closed subgroup, then

$$H_2^{\mathsf{cont}}(\mathcal{G},\mathbb{Z}/p) \longrightarrow H_2^{\mathsf{cont}}(\mathcal{G}/\mathcal{H},\mathbb{Z}/p) \longrightarrow \frac{\mathcal{H} \cap \overline{[\mathcal{G},\mathcal{G}]\mathcal{G}^p}}{\overline{[\mathcal{H},\mathcal{G}]\mathcal{H}^p}} \longrightarrow 0$$

Sketch of the proof for $R = \mathbb{Z}/p = \mathbb{F}_p$.

- **Theorem**(Nikolov, Segal, 2007, Ann. of Math.) Let $\mathcal{G}$ be a finitely generated profinite group and $\mathcal{H}$ be a normal closed subgroup. Then $[\mathcal{H},\mathcal{G}]$ and $[\mathcal{H},\mathcal{G}]\mathcal{H}^p$ are closed.
- Then the following cokernels coincide

$$\begin{tikzcd}
H_2^{\mathsf{disc}}(\mathcal{G},\mathbb{Z}/p) \arrow[r] \arrow[d, "\varphi_2"] & H_2^{\mathsf{disc}}(\mathcal{G}/\mathcal{H},\mathbb{Z}/p) \arrow[r] \arrow[d, "\varphi_2"] & Q^{\mathsf{disc}} \arrow[r] \arrow[d, "\cong"] & 0 \\
H_2^{\mathsf{cont}}(\mathcal{G},\mathbb{Z}/p) \arrow[r] & H_2^{\mathsf{cont}}(\mathcal{G}/\mathcal{H},\mathbb{Z}/p) \arrow[r] & Q^{\mathsf{cont}} \arrow[r] & 0
\end{tikzcd}$$

- If $\mathcal{G}$ is a finitely generated pro-p group and $\mathcal{G} = \mathcal{F}/\mathcal{R}$ is its pro-p-presentation, then

$$H_2^{\mathsf{disc}}(\mathcal{F},\mathbb{Z}/p) \longrightarrow H_2^{\mathsf{disc}}(\mathcal{G},\mathbb{Z}/p) \longrightarrow H_2^{\mathsf{cont}}(\mathcal{G},\mathbb{Z}/p) \longrightarrow 0.$$

- We need to find a group $\mathcal{G}$ such that the kernel of

$$H_2^{\mathsf{disc}}(\mathcal{G},\mathbb{Z}/p) \xrightarrow{\varphi_2} H_2^{\mathsf{cont}}(\mathcal{G},\mathbb{Z}/p)$$

is uncountable.

Sketch of the proof for $R = \mathbb{Z}/p = \mathbb{F}_p$.

- The following map is well defined
$$\mathbb{Z}_p \longrightarrow \mathbb{F}_p[[x]], \qquad \alpha \mapsto (1+x)^\alpha,$$
where $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^i$ is the group of p-adic integers.
- We take the pro-p-completion of the double version of p-lamplighter group
$$\mathcal{G} = \mathbb{F}_p[[x]]^2 \rtimes \mathbb{Z}_p.$$
- Using the spectral sequence of the extension we obtain

$$\begin{tikzcd}
\mathbb{F}_p[[x]] \otimes_{\mathbb{F}_p[\mathbb{Z}_p]} \mathbb{F}_p[[x]] \arrow[r] \arrow[d, tail] & \mathbb{F}_p[[x]] \arrow[d, tail] \\
H_2^{\mathsf{disc}}(\mathcal{G}, \mathbb{Z}/p) \arrow[r] & H_2^{\mathsf{cont}}(\mathcal{G}, \mathbb{Z}/p)
\end{tikzcd}$$

- It is enough to prove that the kernel of the map
$$\mathbb{F}_p[[x]] \otimes_{\mathbb{F}_p[\mathbb{Z}_p]} \mathbb{F}_p[[x]] \longrightarrow \mathbb{F}_p[[x]]$$
is uncountable.

Sketch of the proof for $R = \mathbb{Z}/p$.

- In order to proof that the kernel of

$$\mathbb{F}_p[[x]] \otimes_{\mathbb{F}_p[\mathbb{Z}_p]} \mathbb{F}_p[[x]] \longrightarrow \mathbb{F}_p[[x]]$$

  is uncountable, we need the following lemma.
- **Lemma**. Let $\mathbb{F}_p((x))$ be the field of Laurent power series and K be the subfield generated by the image of $\mathbb{Z}_p$. Then $[\mathbb{F}_p((x)) : K]$ is uncountable.
- In order to prove this lemma we consider $\mathbb{F}_p[[x]]$ as a complete metric space and use the **Baire theorem** about countable unions of nowhere dense subsets.
- We use the theory of profinite groups, field extensions and metric spaces.

Sketch of the proof for $R = \mathbb{Q}$.

- The **nonabelian tensor square** of G is the group $G \otimes G$ generated by elements $g \otimes h$ and relations of "nonabelian bilinearity"

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h),$$

$$g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h').$$

- The **nonabelian exterior square**: $G \wedge G := (G \otimes G)/g \otimes g$.
- There is a short exact sequence

$$0 \longrightarrow H_2(G) \longrightarrow G \wedge G \longrightarrow [G, G] \longrightarrow 1.$$

Sketch of the proof for $R = \mathbb{Q}$.

- For two elements a, b of a Lie algebra L we denote by $[a,_n b]$ the Engel commutator:

$$[a,_0 b] = a, \qquad [a,_{n+1} b] := [[a,_n b], b].$$

- For any two elements a, b of any Lie algebra the following holds:

$$[[[a, b], b], a] = [[[a, b], a], b],$$

and the following generalisation holds for any $n \geq 1$:

$$[[a,_{2n} b]\ ,\ a] = \left[\sum_{i=0}^{n-1} (-1)^i [[a,_{2n-1-i} b], [a,_i b]]\ ,\ b\right].$$

Sketch of the proof for $R = \mathbb{Q}$.

- Using these relations we construct a set Θ of concrete elements (indexed by sequences $\{0,1\}^{\mathbb{N}}$) in the kernel
$$H_2(\hat{F}_{\mathbb{Z}}, \mathbb{Z}) = \mathrm{Ker}(\hat{F}_{\mathbb{Z}} \wedge \hat{F}_{\mathbb{Z}} \to [\hat{F}_{\mathbb{Z}}, \hat{F}_{\mathbb{Z}}]),$$
where $F = F(a, b)$.
- We consider the map to the integral version of the lamplighter group
$$F \twoheadrightarrow G = \mathbb{Z}[\mathbb{Z}] \rtimes \mathbb{Z}.$$
- We prove that the image of Θ under the following map is uncountable
$$H_2(\hat{F}_{\mathbb{Z}}, \mathbb{Z}) \longrightarrow H_2(\hat{G}_{\mathbb{Q}}, \mathbb{Q}).$$
- Then the image of the following map is uncountable
$$H_2(\hat{F}_{\mathbb{Z}}, \mathbb{Z}) \to H_2(\hat{F}_{\mathbb{Q}}, \mathbb{Q}).$$
- We use identities in Lie algebras, noncommutative tensor square and we use that $\mathbb{Q}[\mathbb{Q}]$ is countable (in the mod-p case the analogue of this is $\mathbb{F}_p[\mathbb{Z}_p]$).

HR-localization of a group

- A homomorphism $G \to H$ is **2-connected** over R if

$$H_1(G,R) \xrightarrow{\cong} H_1(H,R), \qquad H_2(G,R) \twoheadrightarrow H_2(H,R).$$

- **Theorem** (Stallings, 1965). If $G \to H$ is 2-connected over R, then

$$\hat{G}_R \cong \hat{H}_R.$$

- HR**-localisation** of G is the terminal 2-connected homomorphism over R:

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "\text{2-connected}"'] & G_{HR} \\ & H \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

- Bousfield proved that the HR-localization exists for any group and

$$\pi_1(X_R) = \pi_1(X)_{HR},$$

where X_R is the R-homological localization of a space X.

HR-localisation of a group

- $H\mathbb{Z}$-localisation is closely related to Levine's algebraic closure of groups, which was invented in order to define transfinite Milnor invariants of links.
- Farjoun, Orr and Shelah proved that $H\mathbb{Z}$-localization can be defined as an "infinite version" of Levine's algebraic closure (systems with infinite number of equations).
- $\gamma_\alpha^{\mathbb{Z}}(G) = \gamma_\alpha(G)$ the transfinite lower central series.
- For $R \in \{\mathbb{Z}, \mathbb{Z}/p, \mathbb{Q}\}$ we denote by $\gamma_\alpha^R(G)$ the transfinite lower R-central series of G :

$$\gamma_{\alpha+1}^R = \mathrm{Ker}(\gamma_\alpha^R \to \gamma_\alpha^R/[\gamma_\alpha^R, G] \otimes R).$$

HR-localisation of a group

- There is a natural homomorphism

$$G_{HR} \to \hat{G}_R.$$

- A group G is said to be HR-good if $G_{HR} \cong \hat{G}_R$.
- Bousfield proved the following.
  1. $\exists \alpha$ such that $\gamma_\alpha^R(G_{HR}) = 1$.
  2. If G is finitely generated, $G_{HR}/\gamma_\omega^R = \hat{G}_R$.
  3. If G is f.g., there is an exact sequence

$$H_2(G,R) \to H_2(\hat{G}_R, R) \to G_{HR}/\gamma_{\omega+1}^R \to \hat{G}_R \to 0$$

  4. Moreover, G is HR-good iff the map

$$H_2(G,R) \to H_2(\hat{G}_R, R)$$

     is an epimorphism.
- The result $H_2(\hat{F}_R, R) \neq 0$ is equivalent to the fact that F is HR-bad.

Prim. Motiv.': The Bousfield problem.

- **The Bousfield problem:** Let G be finitely **presented** and R is a **field** $R = \mathbb{Z}/p, \mathbb{Q}$. Is this true that G is HR-good?
$$G_{HR} \overset{?}{\cong} \hat{G}_R.$$
- **The Bousfield problem:** Is this true that
$$H_2(G, R) \longrightarrow H_2(\hat{G}_R, R)$$
is an epimorphism for $R = \mathbb{Z}/p, \mathbb{Q}$ and finitely presented G?
- **Answer:** Now we know, it is not true for $G = F$.
However:
- **Theorem** (Mikhailov and I, 2014).
It is true if we assume that G is metabelian.
- **Theorem** (I, 2017).
It is true if we assume that G is solvable of finite Prüfer rank.
- **Question:** Is it true for all solvable groups?

HR-length

- For any G there exists α such that

$$\gamma_\alpha^R(G_{HR}) = 1.$$

- $HR\text{-}\mathsf{length}(G) := \min\{\alpha \mid \gamma_\alpha^R(G_{HR}) = 1\}$.
- $H_2(\hat{F}_R, R) \neq 0$ is equivalent to

$$HR\text{-}\mathsf{length}(F) \geq \omega + 1.$$

- **Theorem** (Mikhailov and I, 2016)

$$H\mathbb{Z}\text{-}\mathsf{length}(F) \geq \omega + 2.$$

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