HR-localization and HR-length of a group

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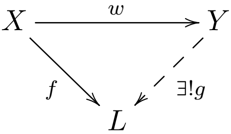
Aim

- General theory of localizations of objects in a category.
- **topology**: theory of homological R-localization.
- **group theory**: HR-localization, HR-length, homology of completions.

Local object

- Let $\mathcal{C}$ be a category and $\mathcal{W} \subseteq \mathrm{Mor}(\mathcal{C})$.
- An object $L \in \mathcal{C}$ is **local** (with respect to $\mathcal{W}$) if for any morphism $w : X \to Y$ in $\mathcal{W}$ and any $f : X \to L$ there exists a unique $g : Y \to L$ such that $gw = f$



- In other words, the induced map is a bijection.

$$w^* : \mathrm{Hom}_{\mathcal{C}}(Y, L) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(X, L)$$

- Roughly speaking, L is local if it can't distinguish morphisms of $\mathcal{W}$ from isomorphisms from its back.
- $\mathsf{Loc}(\mathcal{C}) = \mathsf{Loc}_{\mathcal{W}}(\mathcal{C})$ is the full subcategory of local objects.

Example: local objects in Ab

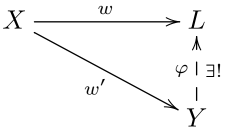
- Let $\mathcal{C} = \mathsf{Ab}$ be the category of abelien groups and $\mathcal{W}$ consists of homomorphisms $w : A \to B$ such that

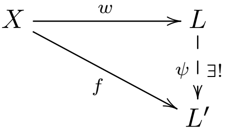
$$w \otimes \mathbb{Q} : A \otimes \mathbb{Q} \xrightarrow{\cong} B \otimes \mathbb{Q}$$

  is an isomorphism.

- $$\mathsf{Loc}(\mathsf{Ab}) = \{\mathbb{Q}\text{-vector spaces}\}.$$

Localisation of an object

- A **localization** of $X \in \mathcal{C}$ is a morphism
$$w : X \to L,$$
where $w \in \mathcal{W}$ and L is local.
- If $w : X \to L$ is a localization then is satisfies two universal properties:
    1. For any $w' : X \to Y$ from $\mathcal{W}$ there exists a unique $\varphi : Y \to L$ such that


    2. For any $f : X \to L'$, where L' is local, there exists a unique $\psi : L \to L'$ such that


- If localisation exists, it is unique up to isomorphism.

General localizations

- **Example.** If $\mathcal{C} = \mathsf{Ab}$ and $\mathcal{W}$ as before, then $A \to A \otimes \mathbb{Q}$ is the localization.
- **Assume** that any object of $\mathcal{C}$ has a localization. Then it defines a functor of localization

$$\mathsf{L} : \mathcal{C} \to \mathsf{Loc}(\mathcal{C}).$$

- The functor L is the left adjoint to the embedding $\mathsf{Loc}(\mathcal{C}) \hookrightarrow \mathcal{C}$.
- The functor L is the localization of the category $\mathcal{C}$ by $\mathcal{W}$

$$\mathcal{C}[\mathcal{W}^{-1}] \cong \mathsf{Loc}(\mathcal{C}).$$

R-localization of a space

- Let R be a commutative ring and $\mathcal{H}$ be the homotopy category of spaces (topological spaces or simplicial sets).
- $\mathcal{W}_R$ the class of R-**homology equivalences** i.e. maps $f : X \to Y$ in $\mathcal{H}$ such that the induced map

$$H_*(f,R) : H_*(X,R) \xrightarrow{\cong} H_*(Y,R)$$

  is an isomorphism.
- A space is R-**local** if it is local with respect to $\mathcal{W}_R$.
- R-localization of a space X is the $\mathcal{W}_R$-localization. It always exists (complicated theorem of Bousfield!) and defines a functor

$$\mathcal{H} \longrightarrow \mathcal{H}_R, \qquad X \mapsto X_R,$$

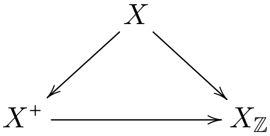
  where $\mathcal{H}_R$ is the subcategory of R-local spaces.
- If R is a subring of $\mathbb{Q}$ and X is 1-connected, then X_R coincides with the Sullivan localization.

Example: plus construction

- Let X be a space and $N \subseteq \pi_1(X)$ be a normal perfect subgroup.
- **Plus construction** of X with respect to N is the universal map in the homotopy category

$$X \longrightarrow X^+,$$

  such that $N \subseteq \mathrm{Ker}(\pi_1(X) \to \pi_1(X^+))$.
- $\pi_1(X^+) = \pi_1(X)/N$ and $H_*(X, \mathbb{Z}) \cong H_*(X^+, \mathbb{Z})$.
- Then there is a triangle of integral homology equivalences



Example: plus construction

- Let A be a ring, $\mathrm{GL}(A) = \varinjlim \mathrm{GL}_n(A)$ and $N = \mathrm{E}(A) = \varinjlim \mathrm{E}_n(A)$.
- Quillen's K-theory is defined as follows

$$K_i(A) = \pi_i(B\mathrm{GL}(A)^+).$$

- It follows from results of F. Keune and Bousfield-Kan that

$$B\mathrm{GL}(A)^+ = B\mathrm{GL}(A)_{\mathbb{Z}}.$$

Fundamental group of R-localization

- Further we assume $R = \mathbb{Z}/n$ or $R \subseteq \mathbb{Q}$.
- If $\tilde{R}$ is a commutative ring and $R \subseteq \tilde{R}$ is the maximal subring of this form, then $\mathcal{W}_R = \mathcal{W}_{\tilde{R}}$.
- **Question:** How to compute $\pi_1(X_R)$?

$$\pi_1(X_R) \cong \pi_1(X)_{HR},$$

where G_{HR} is the HR-**localization** of a group G.

HR-localization of a group

- Let $\mathcal{C} = \mathsf{Gr}$ and $\mathcal{W}_{HR}$ consists of homomorphisms $f : G \to H$ such that
  1. $f_* : H_1(G, R) \xrightarrow{\cong} H_1(H, R)$ is an isomorphism;
  2. $f_* : H_2(G, R) \twoheadrightarrow H_2(H, R)$ is an **epimorphism**.
- Then the HR-**localization** of a group is the $\mathcal{W}_{HR}$-localization. It always exists.

$$G \longrightarrow G_{HR}$$

- An R-**central extension** is a central extension $E \twoheadrightarrow G$, whose kernel is an R-module.

Theorem (Bousfield)

The class of HR-local groups is the smallest class of groups containing the trivial group and closed under R-central extensions, products and kernels ($\Rightarrow$ small limits).

- **Example.** (Pro)nilpotent groups are $H\mathbb{Z}$-local.

Pronilpotent completion and $H\mathbb{Z}$-localization

- Let $R = \mathbb{Z}$.
- For a group G denote by $G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \dots$ its lower central series $\gamma_{n+1}(G) = [\gamma_n(G), G]$.
- The pronilpotent completion of G is

$$\hat{G} := \varprojlim G/\gamma_n(G).$$

- $\hat{G}$ is $H\mathbb{Z}$-local, and hence, there is a unique homomorphism

$$G_{H\mathbb{Z}} \to \hat{G}$$

  that commutes with the maps from G.

Theorem (Bousfield)

If G is finitely generated, $G_{H\mathbb{Z}} \to \hat{G}$ is an epimorphism.

- Usually $G_{H\mathbb{Z}}$ and $\hat{G}$ are uncountable groups. But if G is finitely generated, $H_2(G_{H\mathbb{Z}})$ is countable or finite, while $H_2(\hat{G})$ can be uncountable.

Pronilpotent completion and $H\mathbb{Z}$-localization

- Transfinite lower central series $\gamma_\alpha = \gamma_\alpha(G)$ of G is a transfinite sequence of subgroups such that

$$\gamma_{\alpha+1} = [\gamma_\alpha, G], \qquad \gamma_\tau = \bigcap_{\alpha<\tau} \gamma_\alpha$$

for an ordinal α and a limit ordinal τ.
- G is **transfinitely nilpotent** if there is α such that $\gamma_\alpha(G) = 1$.

Theorem

$G_{H\mathbb{Z}}$ is transfinitely nilpotent.
If G is finitely generated, then $\hat{G} = G_{H\mathbb{Z}}/\gamma_\omega$.

- $H\mathbb{Z}\text{-length}(G) := \min\{\alpha \mid \gamma_\alpha(G_{H\mathbb{Z}}) = 1\}$
- If G is finitely generated, $H\mathbb{Z}\text{-length}(G) \le \omega$ iff $G_{H\mathbb{Z}} \cong \hat{G}$.
- $H\mathbb{Z}\text{-length}(G) < \omega$ iff G is prenilpotent.
- There is a recursive transfinite construction of $G_{H\mathbb{Z}}$ with with the number of steps $H\mathbb{Z}\text{-length}(G)$.

$H\mathbb{Z}$-length of a free group

- The **main mystery** of the theory:

$$H\mathbb{Z}\text{-length}(F) = ?,$$

where F is a finitely generated free group (non-cyclic).

- Bousfield has proved that

$$H\mathbb{Z}\text{-length}(F) \geq \omega + 1$$

Theorem (R.Mikhailov, – (2016, not published))

$$H\mathbb{Z}\text{-length}(F) \geq \omega + 2.$$

- `arXiv:1605.08198v2`

Example: Klein bottle group

- $\mathcal{K} = \mathbb{Z} \rtimes \mathbb{Z}$ is the Klein bottle group.
- $\gamma_n(\mathcal{K}) = 2^{n-1}\mathbb{Z} \rtimes 0.$
- $\hat{\mathcal{K}} = \mathbb{Z}_2 \rtimes \mathbb{Z},$
  where $\mathbb{Z}_2 = \varprojlim \mathbb{Z}/2^n$ is the group of 2-adic integers.
- $H_2(\mathcal{K}_{H\mathbb{Z}}) = H_2(\mathcal{K}) = 0, \qquad H_2(\hat{\mathcal{K}}) \cong H_2(\mathbb{Z}_2) \cong \wedge^2 \mathbb{Z}_2.$
- $\mathcal{K}_{H\mathbb{Z}} \ncong \hat{\mathcal{K}} \cong \mathcal{K}_{H\mathbb{Z}}/\gamma_\omega.$
- $H\mathbb{Z}\text{-length}(\mathcal{K}) > \omega.$

Proposition (R. Mikhailov, – (2016, non-published))

$H\mathbb{Z}\text{-length}(\mathcal{K}) = \omega + 1$ and there is a central extension

$$\wedge^2 \mathbb{Z}_2 \rightarrowtail \mathcal{K}_{H\mathbb{Z}} \twoheadrightarrow \mathbb{Z}_2 \rtimes \mathbb{Z}.$$

Example: Klein bottle group

$\wedge^2 \mathbb{Z}_2 \cong \wedge^2 \mathbb{Q}_2$ is a $\mathbb{Q}$-vector space.

Proposition (R. Mikhailov, – (2016, not published))

$$\mathcal{K}_{H\mathbb{Z}} \cong \mathbb{Z} \times \mathbb{Z}_2 \times (\wedge^2 \mathbb{Z}_2)$$

$$(n,a,\alpha)(m,b,\beta) = \left(n+m, \quad a + (-1)^n b, \quad \alpha + \beta + \frac{(-1)^n}{2} \cdot a \wedge b\right)$$

The first complete description of $G_{H\mathbb{Z}}$, for a finitely generated group G, when $G_{H\mathbb{Z}} \ncong \hat{G}$.

Finitely presented groups of the form $M \rtimes C$

- $C = \langle t \rangle$ is the infinite cyclic group.
- M is a finitely generated $\mathbb{Z}[C]$-module.
- (Bieri-Strebel) The group $M \rtimes C$ is finitely presented iff
  1. $V = M \otimes \mathbb{Q}$ is finite dimensional;
  2. the torsion subgroup of M is finite;
  3. there is a generator t of C such that the characteristic polynomial χ_M of $t \otimes \mathbb{Q} \in \mathrm{GL}(V)$ is integral.

Theorem (Mikhailov, -, 2016, non-published)

Let $G = M \rtimes C$ be finitely presented and $\mu_M = (x - \lambda_1)^{m_1} \dots (x - \lambda_l)^{m_l}$ be the minimal polynomial of $t \otimes \mathbb{Q}$, where $\lambda_1, \dots, \lambda_l \in \mathbb{C}$ are distinct.

1. *Assume that $\lambda_i \lambda_j = 1$ holds only if $\lambda_i = \lambda_j = 1$. Then*
$$H\mathbb{Z}\text{-length}(G) \le \omega.$$
2. *Assume that $\lambda_i \lambda_j = 1$ holds only if either $\lambda_i = \lambda_j = 1$ or $m_i = m_j = 1$. Then*
$$H\mathbb{Z}\text{-length}(G) \le \omega + 1.$$

Example

- $M = \mathbb{Z}^2$ and $C = \langle t \rangle$ acts on M by the matrix

$$\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

- $\mu_M = (x+1)^2$
- $\lambda_1 = -1, \ m_1 = 2, \ \lambda_1 \lambda_1 = 1.$
- $H\mathbb{Z}\text{-length}(M \rtimes C) \geq \omega + 2.$

On a problem of Bousfield

- Let $R = \mathbb{Z}/n$, $R \subseteq \mathbb{Q}$.
- There is a notion of R-**completion** of a group $\hat{G}_R$.
- $\hat{G}_{\mathbb{Z}} = \hat{G}$.
- If G is finitely generated, $\hat{G}_{\mathbb{Z}/p}$ is the pro-p-completion.
- We understand $\hat{G}_R$ well. Usually we do not understand G_{HR}.
- **General question:** When $G_{HR} \cong \hat{G}_R$?
- **Bousfield's conjecture:** Let K be a field $\mathbb{Z}/p$ or $\mathbb{Q}$ and G be a finitely **presented** group. Then $G_{HK} \cong \hat{G}_K$.

Theorem (R.Mikhailov, – (2014))

If G is a metabelian finitely presented group and $K = \mathbb{Z}/p$ or $K = \mathbb{Q}$, then

$$G_{HK} \cong \hat{G}_K.$$

On a problem of Bousfield

Theorem (R.Mikhailov, – (2014))

Let G be a finitely presented metabelian group. Then

$$H_2(G,\mathbb{Z}/p) \longrightarrow H_2(\hat{G}_{\mathbb{Z}/p},\mathbb{Z}/p),$$
$$H_2(G,\mathbb{Q}) \longrightarrow H_2(\hat{G}_{\mathbb{Q}},\mathbb{Q}),$$
$$H_2(G,\mathbb{Z}/p) \longrightarrow H_2(\hat{G},\mathbb{Z}/p).$$

are epimorphisms.

- $H^2(\hat{G}_p,\mathbb{Z}/p) = H^2_{cont}(\hat{G}_p,\mathbb{Z}/p)$.
- The cokernel of the map

$$H_2(G) \to H_2(\hat{G})$$

is divisible.

Homology of completions of free groups

- **What do we know about homology of completions of free groups?**
- Bousfield (1977): $H_2(\hat{F})$ is uncountable. There is an epimorphism

$$H_2(\hat{F}) \twoheadrightarrow \mathbb{Q}^{\oplus \mathbf{c}}.$$

- Bousfield (1992): One of two groups $H_2(\hat{F}_p, \mathbb{Z}/p)$, $H_3(\hat{F}_p, \mathbb{Z}/p)$ is uncountable.
- **Questions:**
- $H_2(\hat{F}) \overset{?}{\cong} \mathbb{Q}^{\oplus \mathbf{c}}$
- Weaker question $H_2(\hat{F}, \mathbb{Z}/p) \overset{?}{=} 0$
- $H_3(\hat{F}) \overset{?}{=} 0$
- $H_2(\hat{F}_p, \mathbb{Z}/p)$, $H_2(\hat{F}_{\mathbb{Q}}, \mathbb{Q})$ =?