Rank-one convexity of certain Bellman functions

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July 28, 2014

Abstract

We prove a theorem of Kirchheim and Kristensen about rank-one convexity of a certain Bellman function related to the Ornstein counter-example.

1 Introduction.

Let n, m, and k be natural numbers. By the symbol $\mathrm{Sym}_k(\mathbb{R}^n, \mathbb{R}^m)$ we denote the space of all symmetric k-tensors on $\mathbb{R}^n$ with values in $\mathbb{R}^m$. In other words, $\mathrm{Sym}_k(\mathbb{R}^n, \mathbb{R}^m)$ consists of all k-linear symmetric mappings from $(\mathbb{R}^n)^k$ to $\mathbb{R}^m$. If φ is a k times differentiable function from $\mathbb{R}^n$ to $\mathbb{R}^m$, then its k-th differential, denoted by $D^k[\varphi]$, can be considered as a function from $\mathbb{R}^n$ to $\mathrm{Sym}_k(\mathbb{R}^n, \mathbb{R}^m)$, i.e. for each x $D^k[\varphi](x)$ is a symmetric k-tensor.

We are going to study some Bellman functions whose cost function evaluates the k-th gradient of the "process". Let $V : \mathrm{Sym}_k(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}$ be a continuous function, homogeneous of order one, i.e. $V(\lambda\xi) = \lambda V(\xi)$ for all $\lambda \in \mathbb{R}_+$[1], and let $\xi \in \mathrm{Sym}_k(\mathbb{R}^n, \mathbb{R}^m)$. Define the class of admissible functions,

$$U_\lambda = C_0^\infty([0, \lambda]^n, \mathbb{R}^m), \tag{1}$$

i.e. U_λ consists of smooth functions with values in $\mathbb{R}^m$ that are supported strictly inside the cube with sidelength λ. We introduce the Bellman function,

$$\mathbb{B}_\lambda(x) = \inf_{\varphi \in U_\lambda} \int\limits_{[0,\lambda]^n} V\big(x + D^k[\varphi](y)\big)\, dy, \quad x \in \mathrm{Sym}_k(\mathbb{R}^n, \mathbb{R}^m). \tag{2}$$

Definition 1. *We say that a function $F : \mathrm{Sym}_k(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}$ is rank-one convex if it is convex in the direction of any rank-one tensor.*

In other words, F is rank-one convex if for any $x \in \mathrm{Sym}_k(\mathbb{R}^n, \mathbb{R}^m)$ and any $\xi \in \mathrm{Sym}_k(\mathbb{R}^n, \mathbb{R}^m)$ that is rank-one (i.e. there exist some $b \in \mathbb{R}^m$ and $a \in \mathbb{R}^n$ such that $\xi = b \otimes a^{\otimes k}$, i.e. $\xi(x_1, \ldots, x_k) = b \cdot (\prod_{i=1}^k \langle x_i, a\rangle)$ for any $x_1, \ldots, x_k \in \mathbb{R}^n$, where the angular brackets denote the scalar product in $\mathbb{R}^n$) the inequality

$$F(x) \leqslant \frac{F(x+\xi) + F(x-\xi)}{2} \tag{3}$$

holds true. The following theorem was mentioned in [1].

Theorem 1. *The function $\mathbb{B}_\lambda$ given by formula (2) is rank-one convex.*

The aim of this note is to provide a proof for this claim. We will give heuristics, and then pass to technicalities. But before it we note two easy things. First, the function $\mathbb{B}_\lambda$ is homogeneous of order one (because the function V is). Second, up to some affine transformations, the function $\mathbb{B}_\lambda$ does not depend on the parameter λ. Indeed, the function φ belongs to U_λ if and only if the

[1] This is not the usual linear homogeneity, here we have the relation $V(\lambda\xi) = \lambda V(\xi)$ only for positive λ.

function φ_λ, $\varphi_\lambda(x) = \varphi(\lambda x)$, belongs to U_1. Moreover, $D^k[\varphi](y) = \lambda^{-k} D^k[\varphi_\lambda](\lambda^{-1}y)$. Therefore, we can write

$$\begin{aligned}
\mathbb{B}_\lambda(x) = \inf_{\varphi \in U_\lambda} \int\limits_{[0,\lambda]^n} V\big(x + D^k[\varphi](y)\big)\,dy = \inf_{\varphi_\lambda \in U_1} \int\limits_{[0,\lambda]^n} V\big(x + \lambda^{-k} D^k[\varphi_\lambda](\lambda^{-1}y)\big)\,dy = \\
\inf_{\varphi_\lambda \in U_1} \int\limits_{[0,\lambda]^n} \lambda^{-k+n} V\big(\lambda^k x + D^k[\varphi_\lambda](\lambda^{-1}y)\big)\,d(\lambda^{-1}y) = \lambda^{-k+n}\,\mathbb{B}_1(\lambda^k x) = \lambda^{-n}\,\mathbb{B}_1(x).
\end{aligned} \tag{4}$$

So, it is enough to prove the theorem for the case $\lambda = 1$. For brevity, we write $\mathbb{B}$ for $\mathbb{B}_1$ and U for U_1.

2 Heuristics.

The ideology of the proof is rather standard and can be found in many papers on the Bellman function method in analysis, e.g. in book [2]. Let φ_x denote an *optimizer* for the Bellman function $\mathbb{B}$ at the point x, i.e. a function belonging to the class U at which the infimum in formula (2) is attained (we skip the question why does such a function exist). Let ξ be a rank-one tensor, by symbols x^+ and x^- we denote the tensors $x+\xi$ and $x-\xi$ correspondingly. To prove the theorem, it is enough to construct a function $\varphi \in U$ such that

$$\int\limits_{[0,1]^n} V\big(x + D^k[\varphi](y)\big)\,dy = \frac{1}{2}\int\limits_{[0,1]^n} V\big(x^+ + D^k[\varphi_{x^+}](y)\big)\,dy + \frac{1}{2}\int\limits_{[0,1]^n} V\big(x^- + D^k[\varphi_{x^-}](y)\big)\,dy. \tag{5}$$

Indeed, by the definition of the optimizers, the expression on the right-hand side is nothing but the right part of inequality (3), whereas the expression on the left-hand side is not bigger than $\mathbb{B}(x)$. To construct such a function φ, we need one more assumption.

Let ℓ_ξ be a function in U such that $D^k[\ell_\xi]$ equals either ξ or $-\xi$. The reader can easily see that such functions do not exist (for example, due to continuity reasons), however, functions with almost these properties exist. So, we assume that ℓ_ξ exists. Let Ω_+ be the set where $D^k[\ell_\xi]$ equals ξ, and let Ω_- be the set where this function equals $-\xi$. It is natural to suppose that they have the same area. By our assumptions, $\Omega_+ \cup \Omega_- = [0,1]^n$. We make one more assumption that $\Omega_+ = \cup_{j=1}^{N_+} Q_j^+$ and $\Omega_- = \cup_{j=1}^{N_-} Q_j^-$, where N_+ and N_- are some natural numbers and $Q_j^\pm$ are cubes with centers $y_j^\pm$. We note that this assumption is also a bit illegal even for approximations of ℓ_ξ.

With this ℓ_ξ, which is usually called an *elementary laminate*, we can construct the function φ with ease. We define the functions $\varphi_j^\pm$ by formula

$$\varphi_j^\pm(y) = \Big(\varphi_{x^\pm}\Big)_{|Q_j^\pm|^{-1}}(y - y_j^\pm), \tag{6}$$

i.e. φ_j^+ is the function φ_{x^+} adjusted to the cube Q_j^+ (and similarly with the minus sign); here $|Q_j^\pm|$ stands for the side length of $Q_j^\pm$. Define the function φ by formula

$$\varphi = \ell_\xi + \sum_\pm \sum_{j=1}^{N_\pm} \varphi_j^\pm. \tag{7}$$

We have to verify equality (5). We calculate the integral on each of the partition cubes individually:

$$\begin{aligned}
\int\limits_{Q_j^\pm} V\big(x + D^k[\varphi](y)\big)\,dy = \int\limits_{Q_j^\pm} V\big(x \pm \xi + D^k[\varphi_j^\pm](y)\big)\,dy = \\
\int\limits_{[0,|Q_j^\pm|]^n} V\Big(x^\pm + D^k\Big[\big(\varphi_{x^\pm}\big)_{|Q_j^\pm|^{-1}}\Big](y)\Big)\,dy = \mathrm{vol}_n(Q_j^\pm)\,\mathbb{B}(x^\pm).
\end{aligned} \tag{8}$$

Summing over all the cubes and recalling that $\sum_{j=1}^{N_+} \mathrm{vol}_n(Q_j^+) = \sum_{j=1}^{N_-} \mathrm{vol}_n(Q_j^-) = \frac{1}{2}$ (because the sets Ω_+ and Ω_- have one and the same area), we get equality (5).

3 Rigorous proof.

We have seen that the proof can be naturally divided into two parts: the first one is construction of an approximation of the function ℓ_ξ, the second one is construction of the function φ from it.

3.1 Construction of elementary laminate

We start with building a certain amount of splines, i.e. piecewise polynomial functions on the line.

Lemma 1. *Let s be some natural number and let ε be a positive real number. There exists a C^{s-1}-smooth function $h_{s,\varepsilon}$ on the line supported in $[0,1]$ such that*

$$D^s[h_{s,\varepsilon}](x) = \pm 1 \quad \text{for almost all } x \in [0,1]; \quad \big\|D^{s-1}[h_{s,\varepsilon}]\big\|_{L_\infty(\mathbb{R})} < \varepsilon.$$

Proof. The proof is by induction in s. For the case $s=1$ on can take the function whose derivative equals 1 on each interval $(\frac{2k}{2N}, \frac{2k+1}{2N})$, $k = 0,1,\dots,N-1$ and equals -1 on each interval $(\frac{2k+1}{2N}, \frac{2k+2}{2N})$, $k = 0,1,\dots,N-1$ (the saw-function), here N is bigger than $2\varepsilon^{-1}$. To pass from s to $s+1$, we are going to integrate $h_{s,\varepsilon}$. Indeed, the function $h_{s+1,\varepsilon}$ given by

$$h_{s+1,\varepsilon}(x) = \int\limits_0^x h_{s,\varepsilon}(y)\,dy \tag{9}$$

satisfies all the properties wanted, except for compactness of the support. The function $h_{s+1,\varepsilon}$ will satisfy the required property provided $\int_0^1 h_{s,\varepsilon} = 0$. We will get this condition by a slight modification of the function $h_{s,\varepsilon}$. Consider the function $\tilde{h}_{s,\varepsilon}$ given by

$$\tilde{h}_{s,\varepsilon}(x) = \begin{cases} 2^{-s}h_{s,\varepsilon}(2x), & x \in [0,\frac12]; \\ -2^{-s}h_{s,\varepsilon}(2x-1), & x \in [\frac12, 1]. \end{cases}$$

The function $\tilde{h}_{s,\varepsilon}$ still has all the needed properties, but also has zero integral, which allows to construct $h_{s+1,\varepsilon}$ via formula (9) with $\tilde{h}_{s,\varepsilon}$ instead of $h_{s,\varepsilon}$. $\square$

We note that the sets where $D^{s+1}[h_{s,\varepsilon}]$ has definite sign have measure $\frac12$. Moreover, together with the smallness of the $(s-1)$-th derivative, we get the smallness of all junior derivatives. We also note that the role of the interval $[0,1]$ is not crucial here, one can replace it by any other interval $I \subset \mathbb{R}$. In such a case, we say that the function $h_{s,\varepsilon}$ is modeled on the interval I and denote such a function by $h_{s,\varepsilon,I}$.

The previous lemma is the key ingredient for construction of approximations of ℓ_ξ. Let $\xi = b \otimes a^{\otimes k}$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, be a rank-one tensor. Consider the function $\ell_{\xi,\varepsilon,I}\colon \mathbb{R}^n \to \mathbb{R}^m$ given by formula

$$\ell_{\xi,\varepsilon,I}(x) = b \cdot h_{k,\varepsilon,I}(\langle a, x\rangle), \quad x \in \mathbb{R}^n.$$

This function is not far from the "ideal" laminate, $D^k[\ell_{\xi,\varepsilon,I}]$ equals ξ or $-\xi$ on a strip $\{\langle a,x\rangle \in I\}$, but unfortunately, $\ell_{\xi,\varepsilon,I}$ is not supported in $[0,1]^n$. However, we have the smallness of $D^{k-1}[\ell_{\xi,\varepsilon,I}]$ and compactness of support at the direction a ($\ell_{\xi,\varepsilon,I}$ is supported in a strip $\{\langle a,x\rangle \in I\}$).

Our aim is to cover the cube $[0,1]^n$ by thin strips of the type $\{\langle x,a\rangle \in I\}$ and then smooth out the discontinuity on the intersection of each strip with the boundary of the cube individually. Let I be some interval on the line. By the symbol $S_{a,I}$ we denote the strip $\{x \in \mathbb{R}^n \mid \langle a,x\rangle \in I\}$. Let $a^\perp$ be the orthogonal complement of a, let $\pi_{a^\perp}$ be the orthogonal projection onto this hyperplane.

Lemma 2. *There exists a convex set $I_a^\perp$ in $a^\perp$ such that*

$$I_a^\perp \times I \subset S_{a,I} \cap [0,1]^n, \ \text{but} \ \operatorname{vol}_n\Big(S_{a,I} \cap [0,1]^n \setminus I_a^\perp \times I\Big) \leqslant C(a)|I|^2.$$

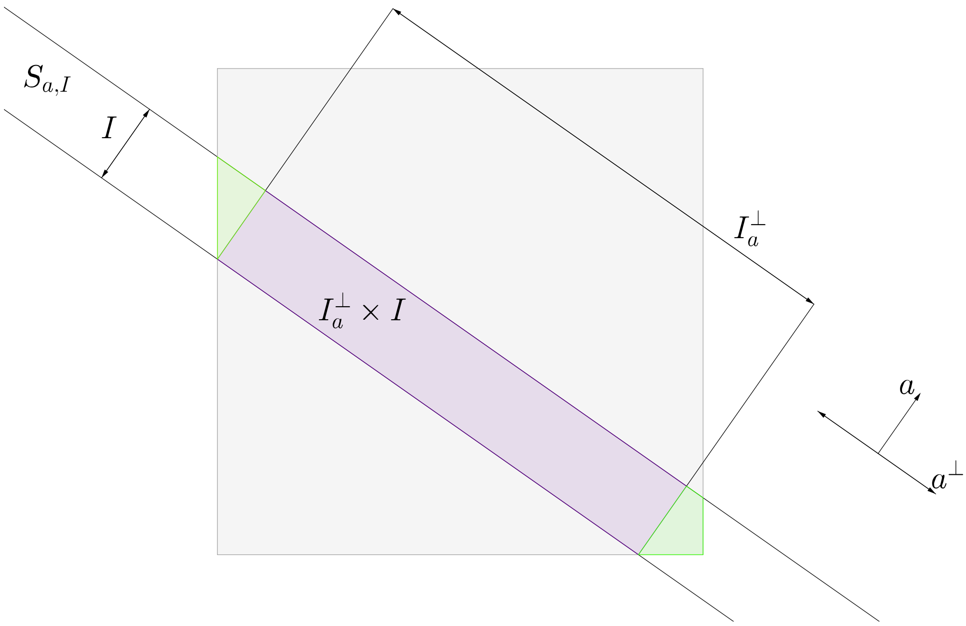

Figure 1: Illustration to Lemma 2.

In this lemma, the product of two sets is taken in the coordinates $(a, a^\perp)$, i.e.

$$I_a^\perp \times I = \{x \in \mathbb{R}^n \mid \langle x, a\rangle \in I, \pi_{a^\perp} x \in I_a^\perp\}.$$

The gist of this lemma is that the slice $S_{a,I} \cap [0,1]^n$ is almost an orthogonal product of I and some convex set $I_a^\perp$, provided I is small enough. Figure 1 can make the text easier. The set whose volume we estimate is marked with green.

Proof. We can describe the set $I_a^\perp$ as the maximal by inclusion subset of $a^\perp$ among the subsets A for which the product $A \times I$ lies in $S_{a,I} \cap [0,1]^n$. With such a definition, $I_a^\perp$ is easily seen to be convex. Indeed, if x and y belong to $I_a^\perp$, then the segments $I_x = \{\pi_{a^\perp} x\} \times I$ and $I_y = \{\pi_{a^\perp} y\} \times I$ belong to $[0,1]^n$. If z is some convex combination of x and y, then the corresponding segment $\{\pi_{a^\perp} z\} \times I$ is a convex combination (with the same coefficients) of I_x and I_y. The cube $[0,1]^n$ is a convex figure, thus $I_z \subset [0,1]^n$, which means that $z \in I_a^\perp$. Conclusion: we only have to prove the inequality for the volume for the maximal possible set $I_a^\perp$.

Surely, if $y \in S_{a,I} \setminus I_a^\perp \times I$, then $\mathrm{dist}(y, \partial[0,1]^n) < |I|$ (because there is some point in the segment I_y that lies outside $[0,1]^n$). Moreover, this means that the segment I_y crosses some face of $[0,1]^n$ transversally. So, the set $S_{a,I} \setminus I_a^\perp \times I$ lies inside the set

$$S_{a,I} \cap \big(\cup_j \,(F_j + B_{|I|}(0))\big), \quad F_j \text{ is a face of } [0,1]^n,\ F_j \nparallel a.$$

The volume of this set does not exceed $C(a)|I|^2$ (because it is a finite union of intersections of two stripes of width $|I|$, the non-zero angle between which depends only on a). ☐

Let $\Phi_{a,I}^\delta : a^\perp \to \mathbb{R}$ be a function adapted to the convex set $I_a^\perp$, i.e. a C^∞-smooth function supported in $I_a^\perp$ whose values are in $[0,1]$ and which equals one on a convex set of measure at least $(1-\delta)\operatorname{vol}_{n-1}(I_a^\perp)$ (thus, its derivatives vanish on the interior of the same set, call it $G_{a,I}$). One can easily construct such a function (however, the concavity of the set is required for our argument).

Now we can define the function $L_{\xi,\varepsilon,\delta,\eta}$ which will be the "true" elementary laminate. Let I_j be a partition of $\mathbb{R}$ into intervals of smallness η. We cover the cube $[0,1]^n$ by the strips S_{a,I_j}, construct for them the sets $(I_j)_a^\perp$ and functions Φ^δ_{a,I_j}. Define the function $L_{\xi,\varepsilon,\delta,\eta}$ by formula

$$L_{\xi,\varepsilon,\delta,\eta}(x) = b\cdot\sum_j h_{\xi,\varepsilon,I_j}(\langle a,x\rangle)\Phi^\delta_{a,I_j}(\pi_a^\perp x). \tag{10}$$

Surely, the definition depends on the partition, but we will not use it. For each η, we fix some partition. The following sublemma is nothing but a consequence of the construction.

Sublemma 3. *The function $L_{\xi,\varepsilon,\delta,\eta}\in C^{k-1}([0,1]^n)$ defined by formula (10) possesses the properties listed below.*

1. *It is supported inside the unit cube.*
2. *There exist sets Ω_+ and Ω_- that are finite unions of convex sets, they are of equal volume, which is not less than $\frac{1}{2}(1-\delta)(1-2c(a)\eta\sqrt{n})$, and $D^k[L_{\xi,\varepsilon,\delta,\eta}]$ equals ξ on Ω_+ and $-\xi$ on Ω_-.*
3. *The function $D^k\big[L_{\xi,\varepsilon,\delta,\eta}\big]$ is uniformly bounded by $\|\xi\|+c(\eta,\delta)\varepsilon$.*

Proof. The first property is clear: all the functions summed on the right-hand side of formula (10) are supported in the unit cube. Indeed, for any j the function $h_{\xi,\varepsilon,I_j}(\langle a,x\rangle)\Phi^\delta_{a,I_j}(\pi_a^\perp x)$ is supported on the set $(I_j)_a^\perp\times I_j\subset[0,1]^n$.

The function $D^k[L_{\xi,\varepsilon,\delta,\eta}]$ equals ξ or $-\xi$ on the sets inside $G_{a,I_j}\times I_j$, because $\Phi^\delta_{a,I_j}(\pi_a^\perp x)$ equals 1 and all its derivatives vanish there. By the construction, the volume of each such set is at least $(1-\delta)\operatorname{vol}_n((I_j)_a^\perp\times I_j)$, so the second point would follow from the inequality

$$\sum_j \operatorname{vol}_n((I_j)_a^\perp\times I_j)\geqslant 1-2c(a)\eta\sqrt{n}.$$

But this follows from the volume estimates of Lemma 2:

$$1-\sum_j\operatorname{vol}_n((I_j)_a^\perp\times I_j)=\sum_j\operatorname{vol}_n(S_{a,I_j}\cap[0,1]^n\setminus(I_j)_a^\perp\times I_j)\leqslant c(a)\sum_j|I_j|^2\leqslant\eta c(a)2\sqrt{n},$$

because the common width of the strips that actually intersect the unit cube does exceed $2\sqrt{n}$.

To prove the third point, we differentiate formula (10) k times ($\otimes_{\mathrm{Sym}}$ means symmetrized tensor product):

$$D^k\big[L_{\xi,\varepsilon,\delta,\eta}\big]=\sum_{i=0}^k\sum_j D^i\big[h_{\xi,\varepsilon,I_j}(\langle a,\cdot\rangle)\big]\otimes_{\mathrm{Sym}}D^{k-i}\big[\Phi^\delta_{a,I_j}(\pi_a^\perp\cdot)\big].$$

The summand with $i=k$ results into $\pm\xi\Phi^\delta_{a,I_j}(\pi_a^\perp\cdot)$, which does not exceed $\|\xi\|$ in norm. All the other summands do not exceed $c(\eta,\delta)\varepsilon$, because they include junior derivatives of h, which are bounded by ε by Lemma 1. $\square$

3.2 Construction of optimizer and end of proof.

We are going to prove Theorem 1. We will follow the plot described in Section 2, fixing the inaccuracies in it. The first unclear point is why do optimizers exist. Instead of them, we take almost optimizers. Let ν be a small positive number. By the very definition, there exist functions φ_{x^+} and φ_{x^-} in the class U such that

$$\mathbb{B}(x^\pm)+\nu>\int\limits_{[0,1]^n}V\big(x^\pm+D^k[\varphi_{x^\pm}](y)\big)\,dy.$$

We are going to construct a function φ such that

$$\int\limits_{[0,1]^n}V\big(x+D^k[\varphi](y)\big)\,dy-\nu<\frac{1}{2}\bigg(\int\limits_{[0,1]^n}V\big(x^++D^k[\varphi_{x^+}](y)\big)\,dy+\int\limits_{[0,1]^n}V\big(x^-+D^k[\varphi_{x^-}](y)\big)\,dy\bigg). \tag{11}$$

These inequalities together lead to

$$\mathbb{B}(x) - 2\nu \leqslant \frac{\mathbb{B}(x+\xi) + \mathbb{B}(x-\xi)}{2},$$

which becomes (3) after letting $\nu \to 0$. Let Ω_+ and Ω_- be the sets from Sublemma 3. Each of these two sets can be almost decomposed into two finite disjoint unions of cubes $Q_j^{\pm} \subset \Omega_{\pm}$:

$$\mathrm{vol}_n\left(\Omega_+ \setminus \cup_j Q_j^+\right) = \mathrm{vol}_n\left(\Omega_- \setminus \cup_j Q_j^-\right) < \theta,$$

where θ is as small as we please. Thus Sublemma 3 implies

$$|1 - \mathrm{vol}_n(\cup_j Q_j^{\pm})| \leqslant 1 - (1-\delta)(1 - 2c(a)\eta\sqrt{n}) + 2\theta. \tag{12}$$

We define the function φ by formulas (6) and (7) with $L_{\xi,\varepsilon,\delta,\eta}$ in place of ℓ_ξ. So, recalling calculation (8), we have

$$\int\limits_{\cup_j Q_j^{\pm}} V\big(x + D^k[\varphi](y)\big)\,dy =$$

$$\mathrm{vol}_n(\cup_j Q_j^+) \int\limits_{[0,1]^n} V\big(x^+ + D^k[\varphi_{x^+}](y)\big)\,dy + \mathrm{vol}_n(\cup_j Q_j^-) \int\limits_{[0,1]^n} V\big(x^- + D^k[\varphi_{x^-}](y)\big)\,dy =$$

$$\frac{1}{2}\,\mathrm{vol}_n(\cup_j Q_j^{\pm})\Big(\int\limits_{[0,1]^n} V\big(x^+ + D^k[\varphi_{x^+}](y)\big)\,dy + \int\limits_{[0,1]^n} V\big(x^- + D^k[\varphi_{x^-}](y)\big)\,dy\Big).$$

We claim that the integral over the remaining set (i.e. $[0,1]^n \setminus (\cup_j Q_j^{\pm})$) is small. Indeed, on this set the function $D^k[\varphi]$ coinsides with the function $D^k[L_{\xi,\varepsilon,\delta,\eta}]$ which is bounded in norm there by $\|\xi\| + c(\eta,\delta)\varepsilon$. Choosing η and δ first, and then taking ε to be small, we see that this does not exceed $2\|\xi\|$. We have assumed that the function V is continuous, thus the values of $V\big(x + D^k[\varphi](y)\big)$ are also uniformly bounded. So, an easy estimate gives

$$\Bigg|\int\limits_{[0,1]^n \setminus (\cup_j Q_j^{\pm})} V\big(x + D^k[\varphi](y)\big)\,dy\Bigg| \leqslant (1 - \mathrm{vol}_n(\cup_j Q_j^{\pm})) \sup_{\|\zeta\| \leqslant 2\|\xi\|} |V(\zeta)|.$$

Collecting all the estimates, we get

$$\int\limits_{[0,1]^n} V\big(x + D^k[\varphi](y)\big)\,dy \leqslant \frac{1}{2}\,\mathrm{vol}_n(\cup_j Q_j^{\pm})\Big(\int\limits_{[0,1]^n} V\big(x^+ + D^k[\varphi_{x^+}](y)\big)\,dy +$$

$$\int\limits_{[0,1]^n} V\big(x^- + D^k[\varphi_{x^-}](y)\big)\,dy\Big) + (1 - \mathrm{vol}_n(\cup_j Q_j^{\pm})) \sup_{\|\zeta\| \leqslant 2\|\xi\|} |V(\zeta)|,$$

which proves inequality (11), because $\mathrm{vol}_n(\cup_j Q_j^{\pm}) \to 1$ if $\eta, \delta, \theta \to 0$ by (12).

We have cheated a little: the constructed function φ does not belong to U because it is not smooth and it is supported on $[0,1]^n$, not inside it. The first problem can be fixed, because smooth functions are dense in $C^k([0,1]^n)$ and the second problem can be fixed by a small dilatation. The theorem is finally proved.

References

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