Dorronsoro's theorem and a small generalization.

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Abstract

We give a simple proof of Dorronsoro's theorem (Theorem 2 in [2]) and also use similar ideas to establish some equivalence for embeddings of vector fields.

1 Introduction

Theorem 1 (Dorronsoro's theorem, Theorem 2 in [2]). *For any real-valued function $f \in C_0^\infty(\mathbb{R}^d)$ there exists a real-valued function $F \in \mathcal{H}_1$ such that*

$$I_1[F] \geqslant f; \quad \|F\|_{\mathcal{H}_1} \lesssim \|\nabla f\|_{L_1}.$$

We denote the real Hardy class by $\mathcal{H}_1$ (we address the reader to the book [10] where he can find all the material about the Hardy class $\mathcal{H}_1$ and the BMO space) and the Riesz potential of order a by I_a,

$$I_\alpha[f] = f * c_a|\cdot|^{a-d}, \quad f \in C_0^\infty(\mathbb{R}^d),\ a \in (0,d).$$

Here c_a is the constant such that I_α is the Fourier multiplier with the symbol $|\xi|^{-\alpha}$. Surely, the Riesz potentials may be applied to a function belonging to $\mathcal{H}_1$. Though Theorem 1 may seem a bit sophisticated, we give a corollary that emphasizes its importance.

Corollary 1. $W_1^1(\mathbb{R}^d) \hookrightarrow L_{\frac{d}{d-1},1}(\mathbb{R}^d)$.

Here W_1^1 is the homogeneous Sobolev space, which is the completion of the set C_0^∞ with respect to the norm

$$\|f\|_{W_1^1} = \|\nabla f\|_{L_1}.$$

In what follows, it is convenient to work with complex-valued functions also; we assume that a function in W_1^1 is complex-valued. The symbol $L_{\frac{d}{d-1},1}$ denotes the Lorentz space (see the book [3] for a detailed study of these spaces). The author does not know who was the first to obtain Corollary 1, however, see the paper [5] for even more general (with respect to another interpolation parameter) result. The corollary follows from Theorem 1 if one recalls that the Riesz potential I_1 maps $\mathcal{H}_1$ to $L_{\frac{d}{d-1},1}$ (this may be justified by means of real interpolation).

We give a proof of Theorem 1 in the next section. It differs from the original proof in [2] by two points: it is constructive (i.e. the function F may be computed in terms of f), the original proof used various duality arguments several times; the presented proof may seem more transparent, because we use only some basic facts (such as the Fefferman–Stein theorem or Gustin's boxing inequality) without going into detailed study of fractional maximal functions. However, the machinery that works in our proof is the same as in the original.

In Section 3, we show that in a more general setting, the statements in the style of Theorem 1 are, in fact, equivalent to a proper analog of Gustin's inequality.

Finally, we collect the statements we use without proof in the last section.

2 Proof of theorem 1

We begin with an easy lemma that lies in the heart of all our constructions. By $(-\Delta)^{\frac{1}{2}}$ we denote the Fourier multiplier with the symbol $|\xi|$.

Lemma 1. *For any function $\varphi \in C_0^\infty(\mathbb{R}^d)$, $d \geqslant 2$, the function $(-\Delta)^{\frac{1}{2}}\varphi$ is in $\mathcal{H}_1$.*

Proof. By the Fefferman–Stein theorem, it suffices to verify that $(-\Delta)^{\frac{1}{2}}\varphi \in L_1$ and $R_j[(-\Delta)^{\frac{1}{2}}\varphi]$ is in L_1 for all $j = 1, 2, \dots, d$ (here R_j stands for the Riesz transform). By the very definition,

$$R_j[(-\Delta)^{\frac{1}{2}}\varphi] = \frac{1}{2\pi i}\frac{\partial \varphi}{\partial x_j},$$

which is obviously summable. The function $(-\Delta)^{\frac{1}{2}}\varphi$ needs more study. First, it is a bounded function, because its Fourier transform, which is $\xi \mapsto |\xi|\hat{\varphi}(\xi)$, is summable (it decays rapidly at infinity). Second, it can be rewritten as $I_1[-\Delta\varphi]$, so, outside the support of φ we may integrate by parts:

$$(-\Delta)^{\frac{1}{2}}\varphi(x) = I_1[-\Delta\varphi](x) = -c_1\int\limits_{\mathbb{R}^d}\Delta\varphi(x-t)|t|^{1-d} = -c_1'\int\limits_{\mathbb{R}^d}\varphi(x-t)c|t|^{-1-d}, \quad x \in \operatorname{supp}\varphi,$$

here c_1' denotes some numerical constant that arises from differentiation of the potential. This formula shows that $(-\Delta)^{\frac{1}{2}}\varphi(x) = O(|x|^{-1-d})$, which leads to the desired summability. $\square$

Now fix some hat-function θ (i.e. a $C_0^\infty(\mathbb{R}^d)$-function that is non-negative and equals one on the unit ball). Let R be a positive real number, then $\theta_R(x) = \theta(\frac{x}{R})$. Using Lemma 1 for $\varphi = \theta$ and rescaling, we get a corollary.

Corollary 2. *For any $R > 0$ there exists a real-valued function $\Theta_R \in \mathcal{H}_1$ such that*

$$I_1[\Theta_R] \geqslant \chi_{B_r(0)}; \quad \|\Theta_R\|_{\mathcal{H}_1} \lesssim R^{d-1}$$

uniformly in R.

The symbol χ_ω denotes the characteristic function of a measurable set ω; $B_r(z)$ stands for the ball of radius r centered at z. Specifically, one may take $\Theta_R = (-\Delta)^{\frac{1}{2}}\theta_R$. Obviously, one can change the ball centered at the origin for any other ball of the same radius. So, we have proved Theorem 1 "for the case where f is a characteristic function of a ball". The latter part of the proof is very standard (for example, a similar method leads to the characterisation of measures μ such that $W_1^1 \hookrightarrow L_{\frac{d}{d-1}}(\mu)$, see [7]), the idea is to break the function f into characteristic functions of balls with the control of the W_1^1-norm. For that purpose we need the notion of Hausdorff capacity.

Definition 1. *Let $\alpha \in [0, d]$. The α-Hausdorff capacity of the set $\omega \subset \mathbb{R}^d$ is defined by the formula*

$$H_\infty^\alpha(\omega) = \inf_{\mathcal{B}} \sum r_j^\alpha, \tag{1}$$

where the infimum is taken over all the coverings $\mathcal{B}$ of Ω by closed balls (and the r_j are the radii of the balls).

This notion allows us to prove Theorem 1 "for the case where f is a characteristic function of a set".

Proposition 1. *Let ω be an open subset of $\mathbb{R}^d$. There exists a real-valued function $\Omega \in \mathcal{H}_1$ such that*

$$I_1[\Omega] \geqslant \chi_\omega; \quad \|\Omega\|_{\mathcal{H}_1} \lesssim H_\infty^{d-1}(\omega).$$

To prove the proposition, one simply considers an almost optimal (in formula (1)) covering of ω by the balls $B_{r_j}(x_j)$ and take Ω to be $\sum \Theta_{r_j,x_j}$, where Θ_{r_j,x_j} denotes the function Θ_{r_j} from Corollary 2 adjusted to the ball $B_{r_j}(x_j)$.

Proof of Theorem 1. By using dilations, we may assume that f is supported in a unit cube, and multiplying it by an appropriate scalar, we may assume that $\|\nabla f\|_{L_1} = 1$. For any $j \in \mathbb{Z}_+$, define $\omega_j = \{x \in \mathbb{R}^d \mid f(x) \geqslant j\}$. For each ω_j, we construct a real-valued $\mathcal{H}_1$-function Ω_j such that

$$I_1[\Omega_j] \geqslant \chi_{\omega_j}; \quad \|\Omega_j\|_{\mathcal{H}_1} \lesssim H^{d-1}_{\infty}(\omega_j).$$

Such functions Ω_j exist by virtue of Proposition 1. Define F by the formula

$$F = \sum_{j\geqslant 0} \Omega_j.$$

Then,

$$f \leqslant \sum_{j\geqslant 0} \chi_{\omega_j} \leqslant \sum_{j\geqslant 0} I_1[\Omega_j] = I_1[F].$$

Moreover,

$$\|F\|_{\mathcal{H}_1} \leqslant \sum_{j=0}^{\infty} \|\Omega_j\|_{\mathcal{H}_1} \lesssim \sum_{j=0}^{\infty} H^{d-1}_{\infty}(\omega_j) \lesssim 1 + \int\limits_0^{\infty} H^{d-1}_{\infty}(\{x \in \mathbb{R}^d \mid f(x) \geqslant t\}) \lesssim$$
$$1 + \int\limits_{\mathbb{R}} H^{d-1}(f^{-1}(t)) = 2\|\nabla f\|_{L_1}.$$

Here H^{d-1} denotes the Hausdorff $(d-1)$-measure. The last but one inequality is an application of Gustin's inequality, Theorem 4 (note that, by Sard's theorem, almost all sets $\{x \in \mathbb{R}^d \mid f(x) \geqslant t\}$ have smooth boundary), the last one is the coarea formula. $\square$

3 Embeddings for vector fields

We present a general statement that lies behind Theorem 1. In what follows, let E and F be two finite dimensional vector spaces over $\mathbb{C}$. Consider a function $A : \mathbb{R}^d \times E \mapsto F$ that is a homogeneous polynomial of order m with respect to the first variable and a linear transformation with respect to the second one. In such a case, A matches the differntial operator that maps E-valued vector fields over $\mathbb{R}^d$ to F-valued vector fields by the rule

$$A(\partial)f = \mathcal{F}^{-1}\Big[A(2\pi i\xi)\mathcal{F}[f](\xi)\Big], \quad f : \mathbb{R}^d \to E,$$

the symbol $\mathcal{F}$ denotes the Fourier transform. Surely, the field f must be sufficiently smooth (e.g. belong to the Schwartz class). For example, the differential operator ∇ corresponds to the function A_{∇} given by the formula

$$A_{\nabla}(\xi, e) = \xi e, \quad e \in \mathbb{R}, \xi \in \mathbb{R}^d.$$

Theorem 2 (Van Schaftingen's theorem, [8]). *The inequality*

$$\|\nabla^{n-1} f\|_{L_{\frac{d}{d-1}}} \lesssim \|A(\partial)f\|_{L_1}$$

holds if and only if the polynomial A is elliptic (i.e. $A(\xi, e) = 0$ if and only if $e = 0$ or $\xi = 0$) and cancelling, i.e.

$$\cap_{\xi\in\mathbb{R}^d\setminus\{0\}} A(\xi, E) = \{0\}.$$

Surprisingly, there is no result that is similar to Corollary 1 (this is an open problem whether a similar theorem can be stated with the Lebesgue norm $L_{\frac{d}{d-1}}$ replaced by the Lorentz norm $L_{\frac{d}{d-1},1}$; see the recent survey [9]) in such a general setting. However, we can say something. We need one more definition.

Definition 2. *Let $a \in [0,d)$. If f is a locally summable function on $\mathbb{R}^d$ (or a measure of locally bounded variation), then the fractional maximal operator of order a acts on it by the formula*

$$M_a[f](x) = \sup_{r>0} r^{a-d} \int\limits_{|x-y|\leqslant r} |f|(y)\,dy.$$

So, M_0 is the usual Hardy–Littlewood maximal operator.

Theorem 3. *Let A be as above, let l be any non-zero element of E^*, let $j = 1,2,\dots,d$. The two statements below are equivalent.*

1. *For any smooth compactly supported vector field φ there exists a real-valued function Φ such that*

$$I_1[\Phi] \geqslant \Re\langle \partial_j^{n-1}\varphi, l\rangle; \quad \|\Phi\|_{\mathcal{H}_1} \lesssim \|A(\partial)\varphi\|_{L_1}.$$

2. *For any smooth compactly supported vector field φ and every non-negative Borel measure μ*

$$\Re\int\limits_{\mathbb{R}^d} \langle \partial_j^{n-1}\varphi, l\rangle\,d\mu \lesssim \|A(\partial)\varphi\|_{L_1}\|M_1[\mu]\|_{L_\infty}.$$

Proof. We are going to apply Ky Fan's minimax theorem, Theorem 5. Let X be the unit ball of the BMO space, this set is convex and compact (in the topology $\sigma(\mathrm{BMO},\mathcal{H}_1)$, we use the fact that BMO is dual to $\mathcal{H}_1$). Let Y be given by the formula

$$Y = \{g \in \mathcal{H}_1(\mathbb{R}^d) \mid I_1[g] \geqslant \Re\langle \partial_j^{n-1}\varphi, l\rangle\}.$$

The function $L : X \times Y \to \mathbb{R}$ is defined as follows:

$$L(f,g) = \Re\langle f, g\rangle$$

This function is continuous with respect to the first variable and bilinear. So, by Theorem 5 (we have interchanged the minimum and maximum, we can do this by applying the theorem to the function $-L$, because we are working with a bilinear function L),

$$\max_{f\in X}\min_{g\in Y}\Re\langle f,g\rangle = \min_{g\in Y}\max_{f\in X}\Re\langle f,g\rangle$$

The value on the right-hand side is (by the $\mathcal{H}_1$-BMO duality)

$$\min\{\|g\|_{\mathcal{H}_1} \mid I_1[g] \geqslant \Re\langle \partial_j^{n-1}\varphi, l\rangle\}.$$

So, the first of the two statements listed in Theorem 3 is equivalent to the inequality

$$\max_{f\in X}\min_{g\in Y}\Re\langle f,g\rangle \lesssim \|A(\partial)\varphi\|_{L_1}.$$

Let us calculate the value on the left-hand side (we fix some function f for a while):

$$\min_{g\in Y}\Re\langle f,g\rangle = \min_{g\in Y}\langle I_1[g], \Re(-\Delta)^{\frac12}[f]\rangle.$$

This formula is meaningful, for example, when $I_1[g] \in C_0^\infty$. If $\Re(-\Delta)^{\frac12}[f]$ is not a non-negative distribution, then this minimum equals $-\infty$. Indeed, this follows from Lemma 1: if $\langle \phi, \Re(-\Delta)^{\frac12}[f]\rangle < 0$ for some non-negative C_0^∞-function ϕ, then the value $\langle \Re\langle \partial_j^{n-1}\varphi, l\rangle + \lambda\phi, \Re(-\Delta)^{\frac12}[f]\rangle$ can be as small as we want (and, by Lemma 1, $\Re\langle \partial_j^{n-1}\varphi, l\rangle + \lambda\phi = I_1[g_\lambda]$ for some $g_\lambda \in \mathcal{H}_1$). It is a well-known fact that non-negative distributions are (real-valued non-negative) measures of temperate growth. But if $\Re(-\Delta)^{\frac{1}{2}}[f] = \mu_f$ is a measure, then

$$\min_{g\in Y} \Re\langle f, g\rangle = \min_{g\in Y}\langle I_1[g], \Re(-\Delta)^{\frac{1}{2}}[f]\rangle = \int \Re\langle \partial_j^{n-1}\varphi, l\rangle\, d\mu_f,$$

where μ_f is a non-negative measure of temperate growth such that $\|I_1[\mu_f]\|_{\mathrm{BMO}} \leqslant 1$; this formula is obvious for the case $I_1[g] \in C_0^\infty$, in the other cases it may be obtained by approximation. Thus, by Adams's theorem 6,

$$\max_{f\in X}\min_{g\in Y} \Re\langle f, g\rangle \asymp \max\Big(\big\{\Re\langle \int \partial_j^{n-1}\varphi\, d\mu, l\rangle \;\big|\; \mu \text{ is a non-negative measure such that } \|M_1[\mu]\|_{L_\infty} \leqslant 1\big\}\Big).$$

So, the second statement of Theorem 3 is equivalent to the inequality

$$\max_{f\in X}\min_{g\in Y} \Re\langle f, g\rangle \lesssim \|A(\partial)\varphi\|_{L_1}.$$

$\square$

Theorem 3 shows that statements in the spirit of Dorronsoro's theorem are, in some sense, equivalent to the fact that class of measures μ such that

$$\|\nabla^{n-1} f\|_{L_{\frac{d}{d-1}}(\mu)} \lesssim \|A(\partial)[f]\|_{L_1}$$

does not depend on the operator A.

4 Our tools

Theorem 4 (Gustin's boxing inequality, [4]). *Let ω be an open subset of $\mathbb{R}^d$ with smooth boundary. Then,*

$$H^{d-1}_\infty(\omega) \lesssim H^{d-1}(\partial\omega).$$

Theorem 5 (Ky Fan's minimax theorem). *Let X and Y be convex subsets of linear topological spaces, let X be compact. If a continuous function $L : X \times Y \to \mathbb{R}$ is convex with respect to the first variable and concave with respect to the second one, then*

$$\min_{x\in X}\max_{y\in Y} L(x, y) = \max_{y\in Y}\min_{x\in X} L(x, y)$$

We have stated a simplification of Ky Fan's theorem (for the original version, see the paper [6][1]).

Theorem 6 (Adams's theorem). *Let $a \in (0, d)$ be a fixed number. Then,*

$$\|I_a[f]\|_{\mathrm{BMO}} \lesssim \|M_a[f]\|_{L_\infty}.$$

If f is non-negative and $\int\limits_{\mathbb{R}^d} (1 + |x|)^{-a-d} I_a[f](x)\, dx < \infty$, then

$$\|M_a[f]\|_{L_\infty} \lesssim \|I_a[f]\|_{\mathrm{BMO}}.$$

This theorem was proved in the paper [1].

[1] Our simplification may be an earlier version of the minimax theorem, it may be a result some other mathematician.

References

[1] D. Adams, *A note on Riesz potentials*, Duke Math. J., **42** (1975), 765–778.

[2] J. R. Dorronsoro, *Differentiability properties of functions with bounded variation*, Indiana Univ. Math. Journal, **38**:4 (1989), 1027–1045.

[3] L. Grafakos, *Classical Fourier analysis*, Springer, sec. ed., 2008.

[4] W. Gustin, *Boxing inequalities*, J. Math. Mech., **9**, 229–239.

[5] V. I. Kolyada, *On an embedding of Sobolev spaces*, Mat. Zametki, **54**:3 (1993), 48–71 (Russian).

[6] Ky Fan, *Minimax theorems*, Proc. N.A.S., **39**:1 (1953), 42–47.

[7] V. G. Maz'ya, T. O. Shaposhnikova, *Theory of multipliers in spaces of differentiable functions*, Uspekhi Mat. Nauk, **38**:3(231) (1983), 23–86.

[8] J. van Schaftingen, *Limiting Sobolev inequalities for vector fields and cancelling linear differential operators*, J. of EMS, **15**:3 (2013), 877–921.

[9] J. van Schaftingen, *Limiting Bourgain–Brezis inequalities for systems of linear differential equations: Theme and variations*, J. of fixed point th. and appl. (2014), 1–25.

[10] E. M. Stein, *Harmonic analysis, real-variable methods, orthogonality and oscillatory integrals*, Princeton University press, 1993.